Goal for this week: All of Chapter 3 & some of Chapter 4.

We have seen the equation of a line in $\mathbb{R}^2$,

$$a(x - x_0) + b(y - y_0) = 0$$

and the equation of a plane in $\mathbb{R}^3$,

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

What is "the equation of a line in $\mathbb{R}^3$"? Trick Question!

A line in $\mathbb{R}^3$ <u>cannot</u> be described by a single equation. Instead we have two different ways to describe a line:

- using a parametrization:

$$\vec{r}(t) = (x(t), y(t), z(t))$$

$$= (x_0 + ta, y_0 + tb, z_0 + tc)$$

You can think of

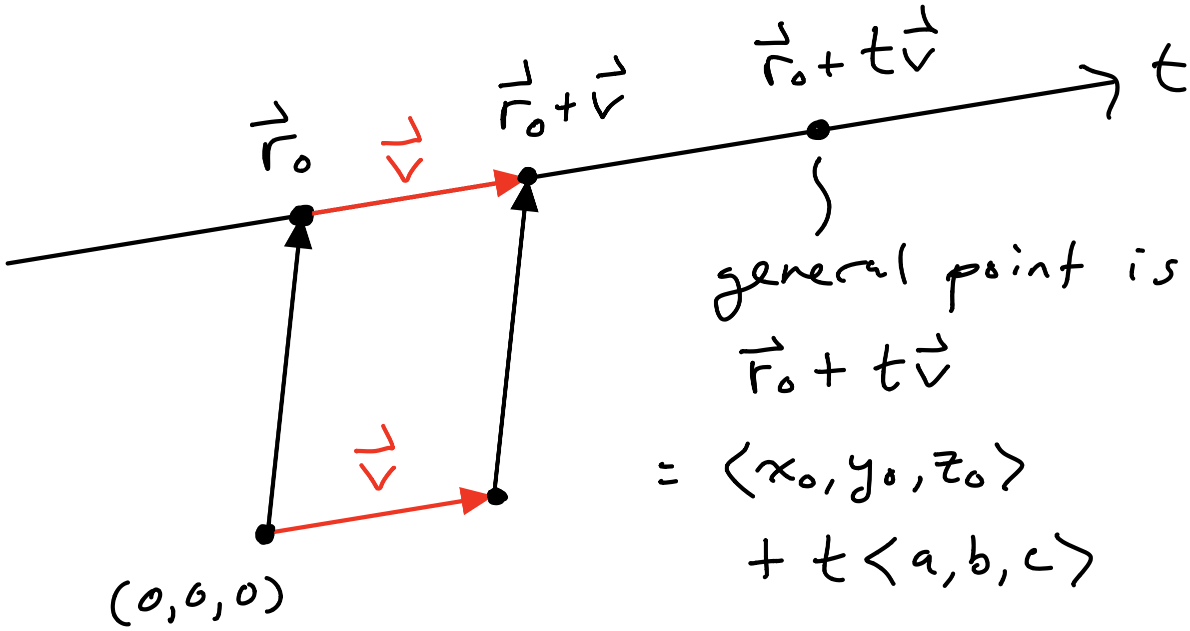
$$\vec{r}_0 = (x_0, y_0, z_0)$$

as the "initial position" of a particle & think of

$$\vec{v} = \langle a, b, c \rangle$$

as a "constant velocity".

Picture:



general point is

$$\vec{r}_0 + t\vec{v}$$

$$= \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle$$

$$= \langle x_0 + ta, y_0 + tb, z_0 + tc \rangle$$

- as an <u>intersection</u> of two (or more) planes:

What happens if we try to "eliminate the parameter"

$$\begin{cases} x = x_0 + ta, \\ y = y_0 + tb, \\ z = z_0 + tc. \end{cases}$$

[These are called the "parametric equations of the line.]

Solve each equation for t:

$$\begin{cases} t = (x - x_0)/a \\ t = (y - y_0)/b \\ t = (z - z_0)/c \end{cases}$$

Combine these to get 3 different equations not involving t:

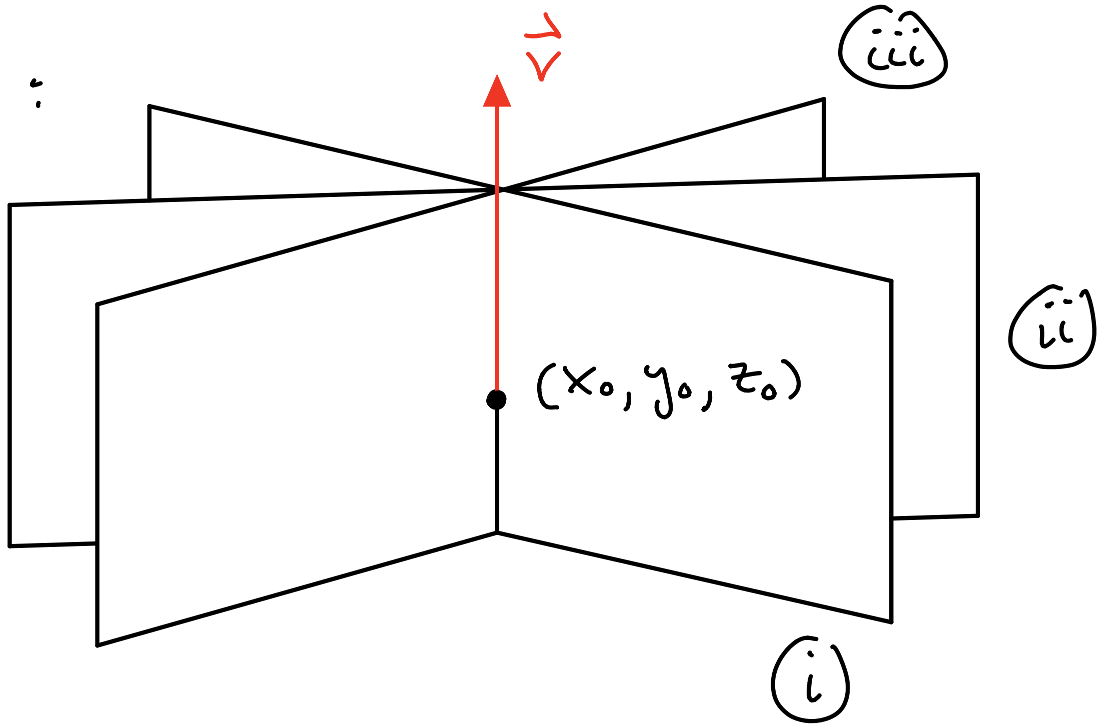
i) $(x-x_0)/a = (y-y_0)/b$

ii) $(x-x_0)/a = (z-z_0)/c$

iii) $(y-y_0)/b = (z-z_0)/c$

Each of these is the equation of a plane in $\mathbb{R}^3$. Any two of these planes intersect at the given line.

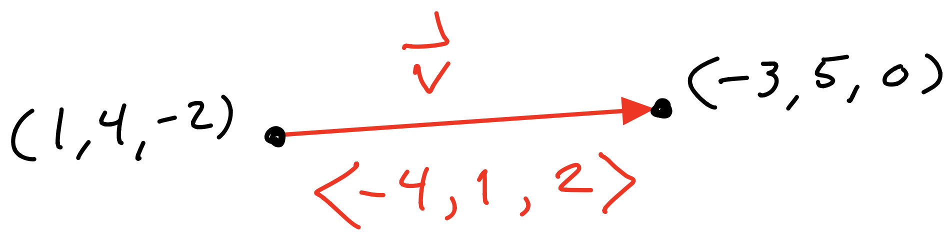
Picture:



The equations i, ii, iii are called the "symmetric equations" of the line.

Example: Find the parametric & symmetric equations of the line in $\mathbb{R}^3$ passing through points

$$P = (1,4,-2) \ \& \ Q = (-3,5,0).$$



If we take $(x_0, y_0, z_0) = P = (1,4,-2)$ & $\vec{v} = \overrightarrow{PQ} = \langle -4, 1, 2\rangle$ then the parametric equations are

$$(x,y,z) = (x_0 + ta, y_0 + tb, z_0 + tc)$$
$$= (1-4t, 4+t, -2+2t)$$

OR $\begin{cases} x = 1-4t, \\ y = 4+t, \\ z = -2+2t. \end{cases}$

By eliminating t we obtain the symmetric equations:

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$$

$$\frac{x-1}{-4} = \frac{y-4}{1} = \frac{z+2}{2},$$

which express the line as the intersection of (any two of) the follwing 3 planes:

i) $(x-1)/(-4) = (y-4)/1$

$$x - 1 = -4y + 16$$

$$x + 4y = 17$$

ii) $(x-1)/(-4) = (z+2)/2$

$$2x - 2 = -4z - 8$$

$$2x + 4z = -6$$

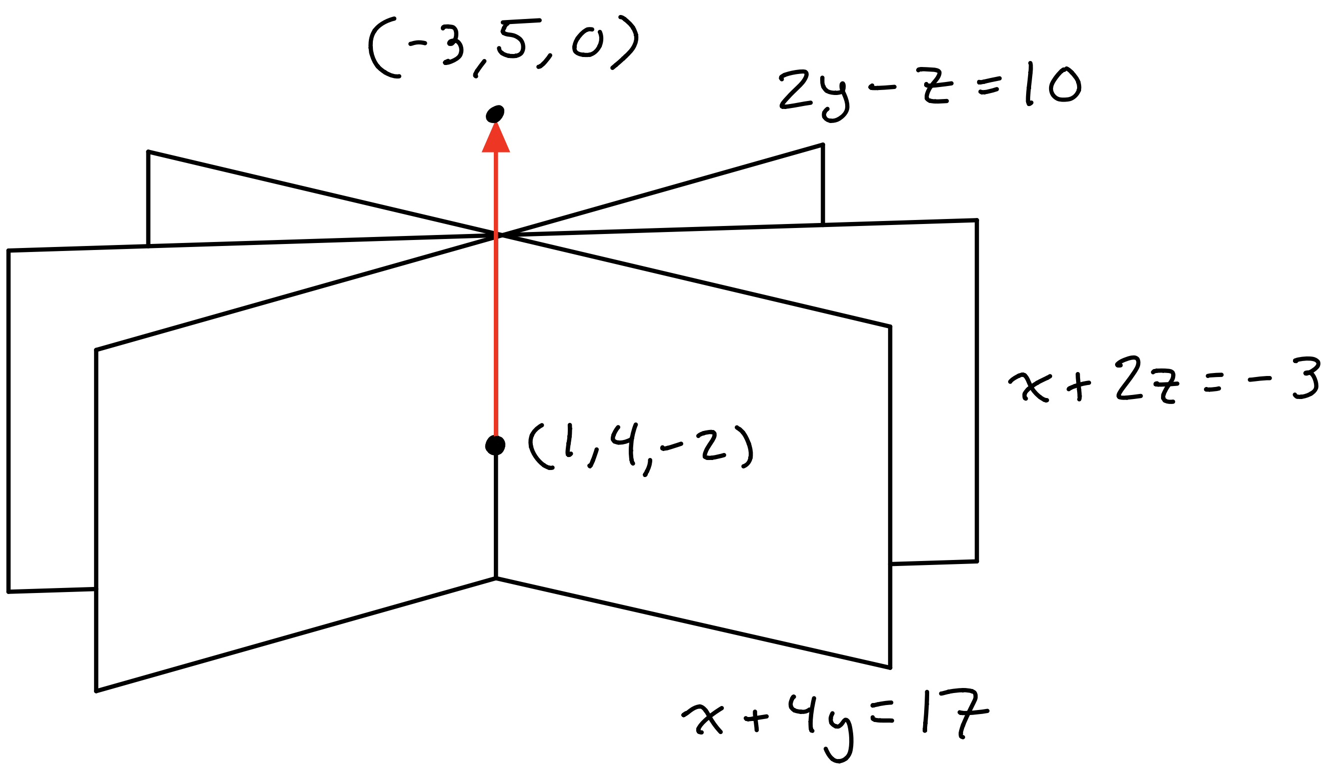
$$x + 2z = -3$$

iii) $(y-4)/1 = (z+2)/2$

$$2y - 8 = z + 2$$

$$2y - z = 10$$

Picture:



Conversely, given any two planes in $\mathbb{R}^3$,

$$\begin{cases} ax + by + cz = d \\ a'x + b'y + c'z = d' \end{cases}$$

we can solve these two equations to obtain the parametrized line of intersection.

Example: $\begin{matrix} ① \\ ② \end{matrix} \begin{cases} x+y+z=4 \\ x+2y+3z=3 \end{cases}$

We will use the method of "elimination". First we replace equation ② by ③ = ② − ①,

$$\begin{array}{rl} & (\cancel{x}+2y+3z=3) \\ - & (\cancel{x}+y+z=4) \\ \hline ③ & \quad y+2z=-1 \end{array}$$

This gives a simpler, but equivalent, system of equations

$$\begin{matrix} ① \\ ③ \end{matrix} \begin{cases} x+y+z=4, \\ \quad\; y+2z=-1. \end{cases}$$

[Say we "eliminated x" from the second equation.]

Next we replace equation ① by ① − ③,

$$
\begin{array}{rl}
 & (x + \cancel{y} + z = 4) \\
- & (0 + \cancel{y} + 2z = -1) \\
\hline
④ & x + 0 - z = 5
\end{array}
$$

Our final equivalent system is

$$
\begin{array}{l}
④ \\
③
\end{array}
\begin{cases}
\boxed{\begin{matrix} x + 0 \\ 0 + y \end{matrix}} \boxed{\begin{matrix} -z \\ +2z \end{matrix}} \begin{matrix} = 5, \\ = -1. \end{matrix}
\end{cases}
$$

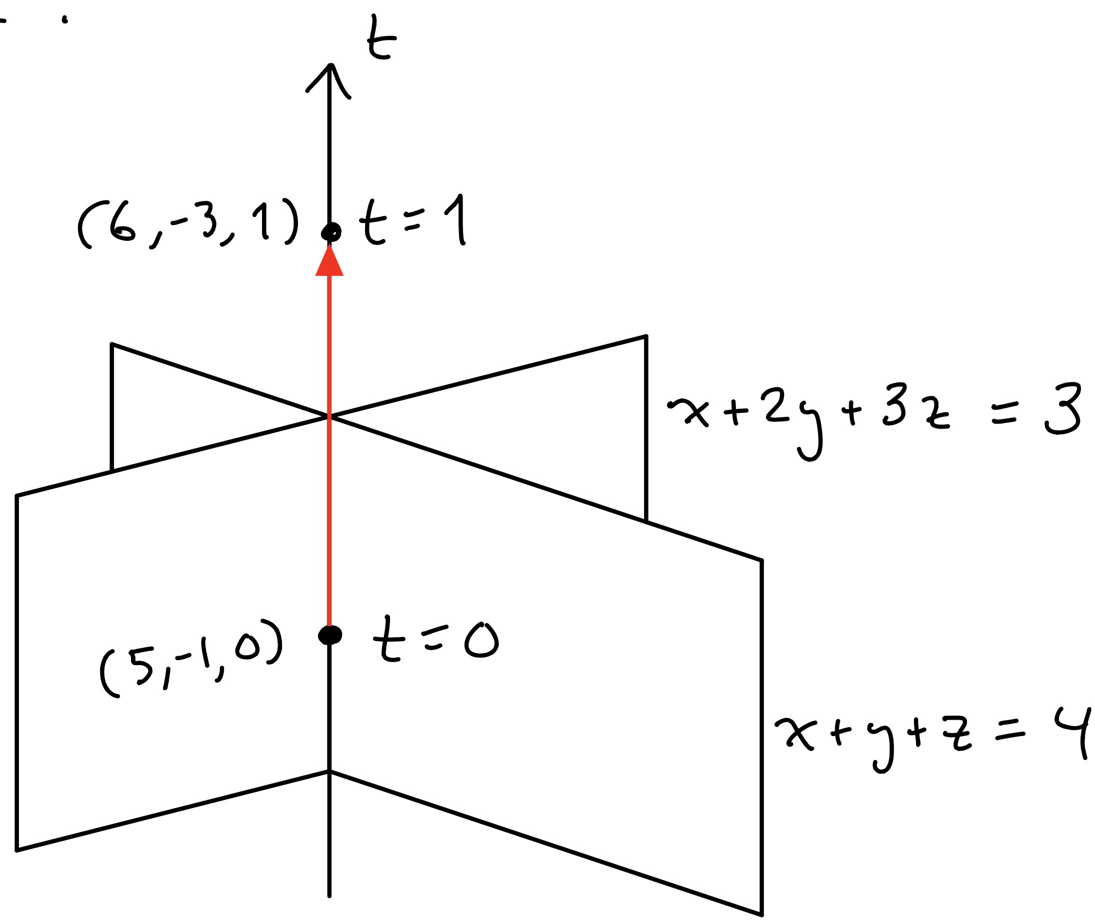
The good thing is that we have solved for the "pivot variables" x & y in terms of the "free variable" z. Let's write down the solution:

$$\begin{cases} x = 5 + z \\ y = -1 - 2z \\ z = 0 + z \end{cases}$$

These look like the parametric equations of a line. I guess we should define $t = z$ (why not?) to get

$$(x, y, z) = (5 + t, -1 - 2t, 0 + t)$$

Picture:



Actually, there is a quicker way to solve this using the cross product.

vectors in the plane $x+y+z=4$ are $\perp$ to the normal vector $\langle 1,1,1\rangle$,

and vectors in the plane $x+2y+3z=3$ are $\perp$ to normal vec $\langle 1,2,3\rangle$.

Therefore every vector in the line of intersection is simultaneously $\perp$ to $\langle 1,1,1\rangle$ & $\langle 1,2,3\rangle$.

The cross product is one such vector:

$$\langle 1,1,1\rangle \times \langle 1,2,3\rangle$$
$$= \langle 1,-2,1\rangle$$

Therefore the line of intersection has the form

$$\begin{cases} x = x_0 + t, \\ y = y_0 - 2t, \\ z = z_0 + t, \end{cases}$$

where (x_0, y_0, z_0) is <u>any</u> point on the line. If we already know such a point then we're done.