Summary of vector arithmetic.

For any vectors $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$ and any scalars $s, t \in \mathbb{R}$, we have the following rules:

- $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
- $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$
- $\vec{u} + \vec{0} = \vec{u}$
- $\vec{u} + (-\vec{u}) = \vec{0}$
- $s(t\vec{u}) = (st)\vec{u}$
- $(s+t)\vec{u} = s\vec{u} + t\vec{u}$
- $s(\vec{u} + \vec{v}) = s\vec{u} + s\vec{v}$
- $1\vec{u} = \vec{u}$ & $0\vec{u} = \vec{0}$
- $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
- $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$
- $t(\vec{u} \cdot \vec{v}) = (t\vec{u}) \cdot \vec{v} = \vec{u} \cdot (t\vec{v})$
- $\|\vec{u}\|^2 = \vec{u} \cdot \vec{u}$

- $\|\vec{u}\| \geq 0$ & $\|\vec{u}\| = 0 \iff \vec{u} = \vec{0}$
- $\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos\theta$

For vectors in $\mathbb{R}^3$ we have another operation called the "cross product". [This <u>only</u> exists in 3D!] Here are the <u>rules</u> for this operation (see pg 169):

- $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$
- $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$
- $t(\vec{u} \times \vec{v}) = (t\vec{u}) \times \vec{v} = \vec{u} \times (t\vec{v})$
- $\vec{u} \times \vec{0} = \vec{0}$
- $\vec{u} \times \vec{u} = \vec{0}$

- $\vec{u} \cdot (\vec{u} \times \vec{v}) = 0$

  $\vec{v} \cdot (\vec{u} \times \vec{v}) = 0.$

Finally, there is one more strange-looking rule:

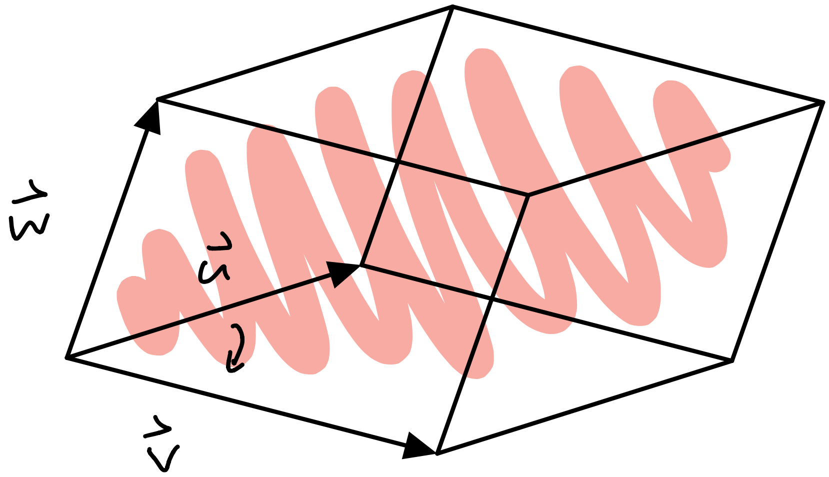
$$\vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w}$$

WHAT ???

This identity comes from a geometric property relating the dot product & cross product:

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = \pm \text{ Volume of the "parallelepiped" spanned by } \vec{u}, \vec{v}, \vec{w}.$$

Here is a picture:



If $\vec{u}, \vec{v}, \vec{w}$ is a "right handed system" then

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = +\text{Volume}$$

and if $\vec{u}, \vec{v}, \vec{w}$ is a "left handed system" then

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = -\text{Volume}.$$

If $\vec{u}, \vec{v}, \vec{w}$ is right handed, then

Right Handed	Left Handed
$\vec{u}, \vec{v}, \vec{w}$	$\vec{u}, \vec{w}, \vec{v}$
$\vec{v}, \vec{w}, \vec{u}$	$\vec{w}, \vec{v}, \vec{u}$
$\vec{w}, \vec{u}, \vec{v}$	$\vec{v}, \vec{u}, \vec{w}$

secretly, this is an aspect of the theory of "determinants"

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

"scalar triple product" "a 3×3 determinant"