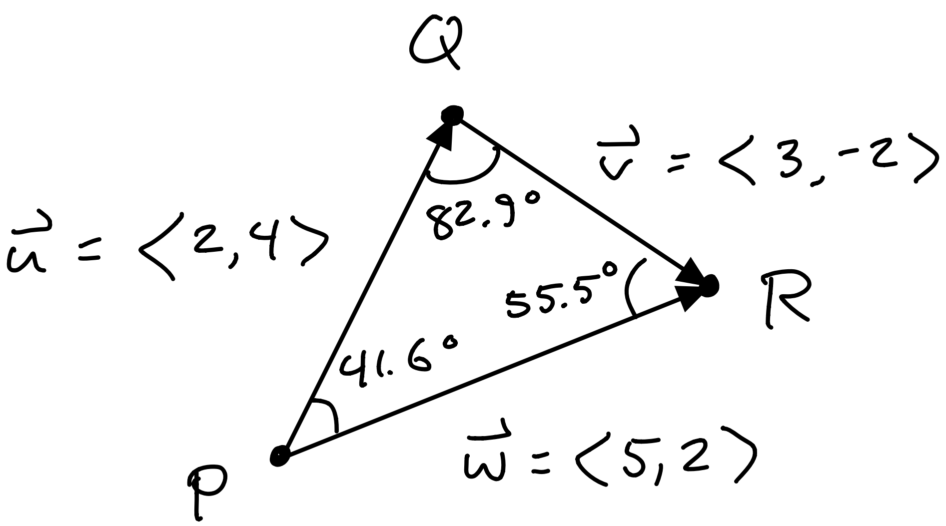
HW 1 due before tomorrow's lecture.
Upload pdf file to Blackboard.
Quiz : 11:40 – 12:00 during
Monday's lecture.

Last time we computed the side lengths & angles of the following triangle in the 2D plane $\mathbb{R}^2$:



we used the formulas

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

$$\|\vec{u}\|^2 = \vec{u} \cdot \vec{u} \quad \text{(Pyth. Thm.)}$$

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \, \|\vec{v}\| \cos\theta$$

[Angle measured between $\vec{u}$ & $\vec{v}$ placed "tail-to-tail".]

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \, \|\vec{v}\|}$$

$$= \frac{\vec{u} \cdot \vec{v}}{\sqrt{\vec{u} \cdot \vec{u}} \, \sqrt{\vec{v} \cdot \vec{v}}}.$$

It turns out that these same formulas work in any number of dimensions. On HW1 Problem 3 you will apply them to the triangle in 3D space $\mathbb{R}^3$ with vertices

$$P = (1, 1, 0)$$

$Q = (1, 0, 2)$

$R = (1, 2, 3).$

Last time we discussed §2.1–2.3

Today: §2.4–2.5.

You are familiar with the equation of a line

$$y = mx + b$$

where m = slope & b = y-intercept.

But this equation <u>doesn't generalize</u> to higher dimensions.

So in this class we prefer to use the form

$$ax + by = c.$$

Even better : We can write this equation as

$$\boxed{a(x-x_0)+b(y-y_0)=0}$$

I claim that this is the equation of the line that :

- passes through point (x_0, y_0)
- is perpendicular to vector $\langle a, b\rangle$

To see why, we make the following observation.

Perpendicular / Orthogonal Vectors.

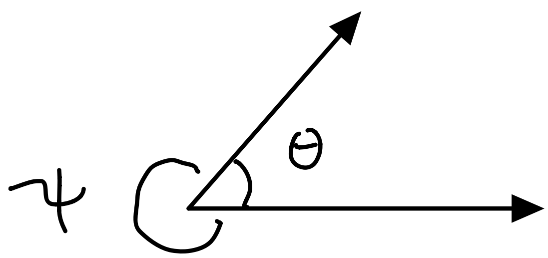
Given two vectors $\vec{u}, \vec{v}$ in any dimensional space :

$$\vec{u} \cdot \vec{v} = 0 \iff \|\vec{u}\|\|\vec{v}\| \cos\theta = 0$$

$$\iff \cos\theta = 0$$

$\iff$ θ is a right angle
($90°$ or $270°$)

[Remark: There are 2 angles between any two vectors, but the have the same cosine.



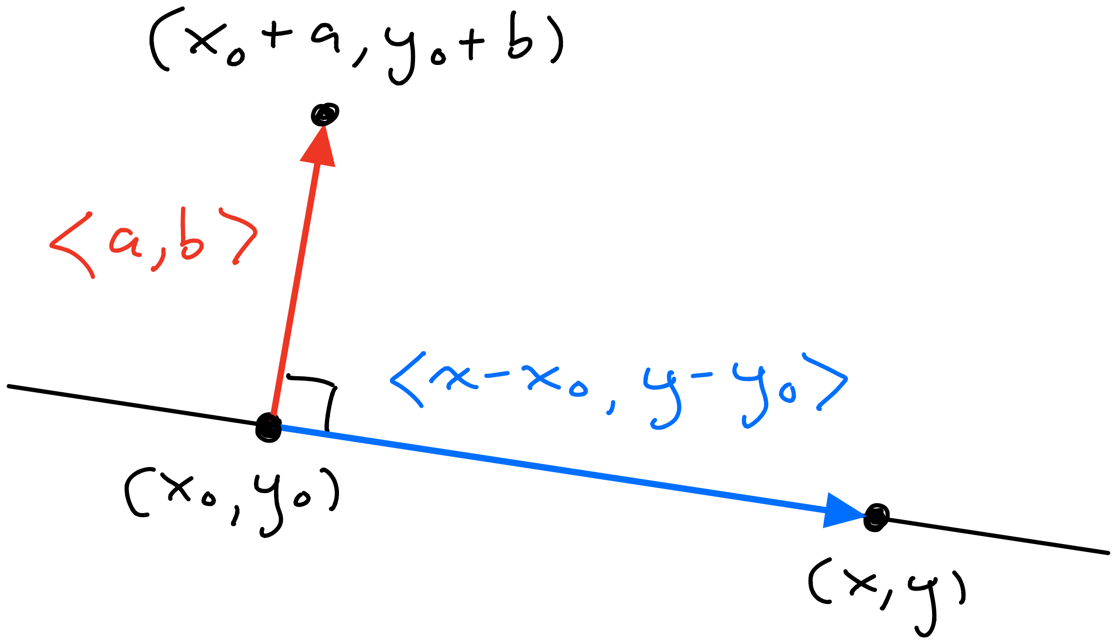
Since $\theta + \psi = 360°$ we have

$$\cos\theta = \cos\psi .$$
]

Summary:

$$\boxed{\vec{u} \circ \vec{v} = 0 \iff \vec{u} \perp \vec{v}}$$

Here is a picture of the desired line:



If (x,y) is a general point on the line then we observe that the vectors $\langle a,b\rangle$ & $\langle x-x_0, y-y_0\rangle$ are perpendicular, so that

$$\langle a,b\rangle \cdot \langle x-x_0, y-y_0\rangle = 0$$

$$a(x-x_0) + b(y-y_0) = 0 \quad \checkmark$$

It is easy enough to convert between this and other forms of the equation such as

$$y = mx + b$$

$$m = (y - y_0)/(x - x_0)$$

$\vdots$

etc.

Example: Find the equation of the line in $\mathbb{R}^2$ passing through point $(1, 2)$ & perpendicular to the vector $\langle 3, 1 \rangle$.

Let $(x_0, y_0) = (1, 2)$

$\langle a, b \rangle = \langle 3, 1 \rangle$

so the equation is

$$a(x - x_0) + b(y - y_0) = 0$$

$$3(x - 1) + 1(y - 2) = 0$$

Rearrange:

$$y - 2 = -3(x - 1)$$

$$y = -3(x-1) + 2$$

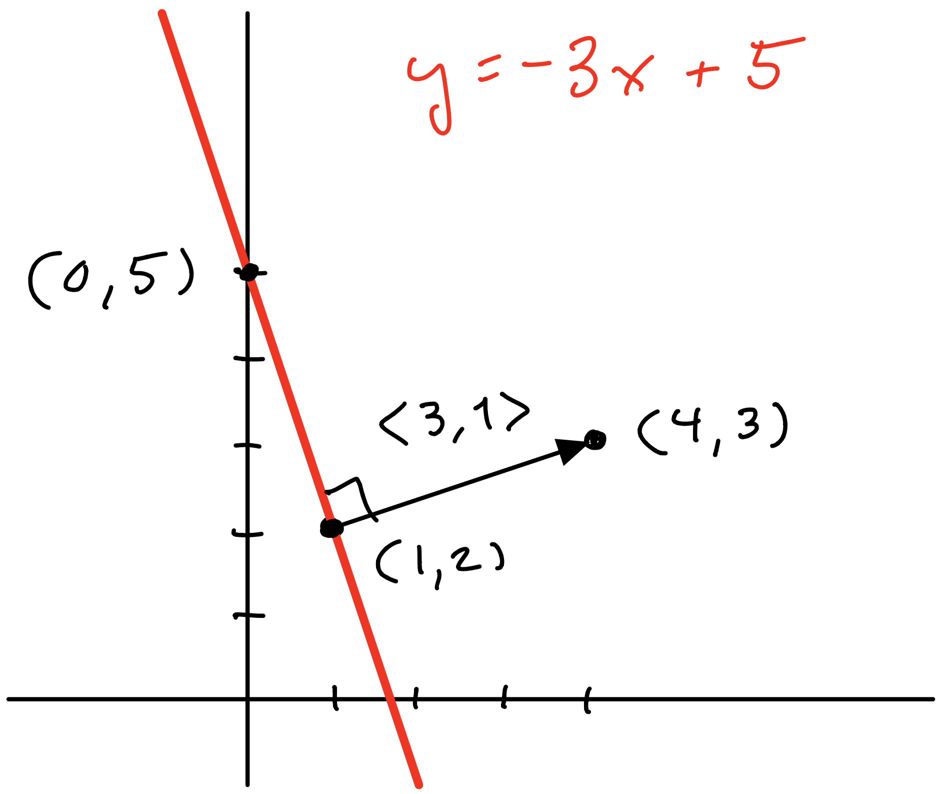
$$y = -3x + 3 + 2$$

$$y = -3x + 5$$

slope $\quad m = -3$

y-intercept $\quad b = +5$

Picture :



The reason that we like the equation

$$a(x-x_0)+b(y-y_0)=0$$

is because it generalizes well to higher dimensions.

For example, consider the equation

$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$$

What shape does this represent?

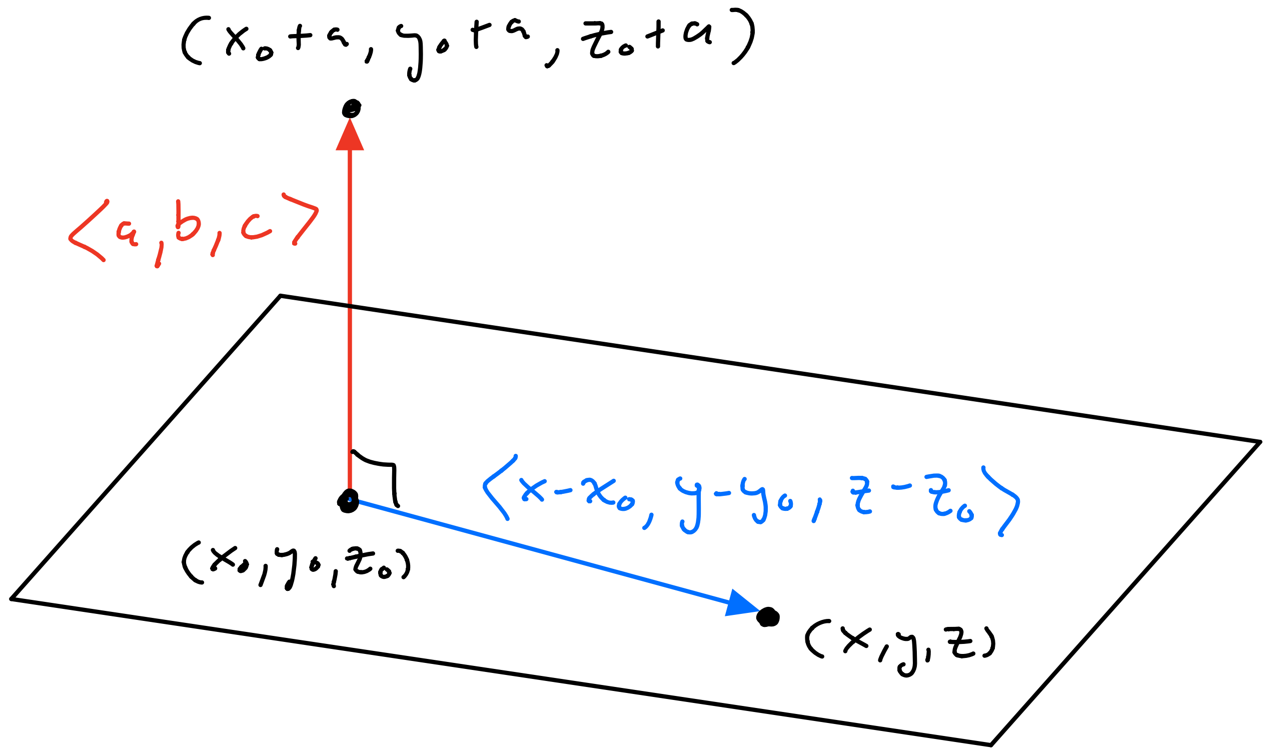
The shape lives in $\mathbb{R}^3$, passes though a point (x_0, y_0, z_0), and is $\perp$ to a vector $\langle a, b, c\rangle$.

So it's a plane!

[Remark: It's <u>not</u> a line in $\mathbb{R}^3$ because a line in $\mathbb{R}^3$ cannot be

defined by a single equation!
A line requires at least 2 equations.]

Picture:



For a general point (x,y,z) in the plane, we observe that vectors $\langle a,b,c \rangle$ & $\langle x-x_0, y-y_0, z-z_0 \rangle$ are perpendicular, so that

$$\langle a,b,c \rangle \cdot \langle x-x_0, y-y_0, z-z_0 \rangle = 0$$

$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0.$$

Sometimes we say $\vec{n}=\langle a,b,c\rangle$ is a "normal vector" to the plane.

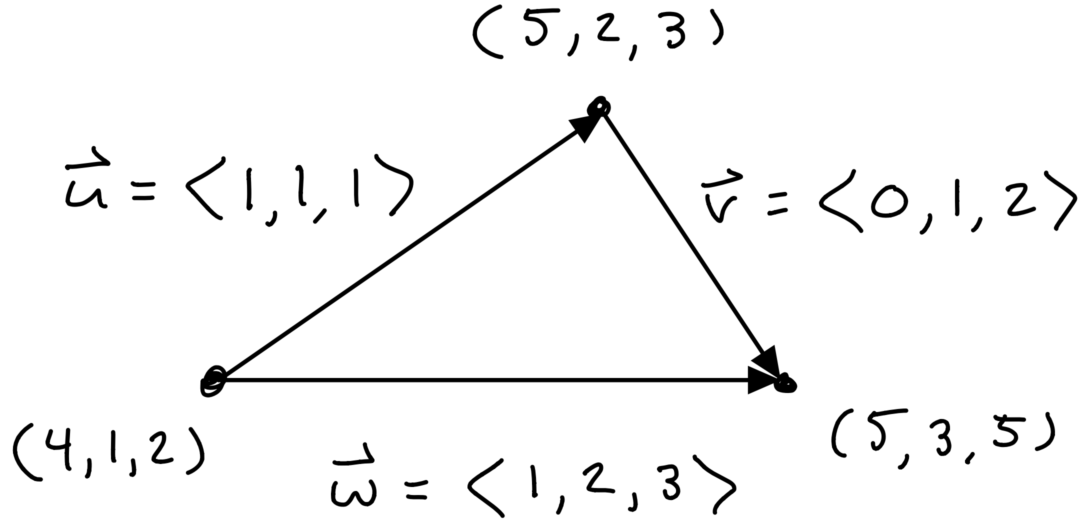
But sometimes we are not given a normal vector & we have to find one ourselves.

Example: Find the equation of the plane passing through the points $P=(4,1,2)$

$Q=(5,2,3)$

$R=(5,3,5)$

We can draw a rough picture of the corresponding triangle:



To find the equation of the plane that contains this triangle, we need a point (x_0, y_0, z_0) and a "normal vector" $\vec{n} = \langle a, b, c \rangle$.

The point is easy:

$$(x_0, y_0, z_0) = \underbrace{P \text{ or } Q \text{ or } R}_{\text{we have 3 perfectly good choices.}}$$

The vector $\vec{n}$ is more difficult, but we have a <u>standard trick</u>.

Definition of Cross Product in $\mathbb{R}^3$:

Given two vectors in $\mathbb{R}^3$

$$\vec{u} = \langle u_1, u_2, u_3 \rangle \quad \& \quad \vec{v} = \langle v_1, v_2, v_3 \rangle$$

we define the cross product vector, which also lives in $\mathbb{R}^3$:

$$\vec{u} \times \vec{v} = \langle u_2 v_3 - u_3 v_2 \, , \, u_3 v_1 - u_1 v_3 \, , \, u_1 v_2 - u_2 v_1 \rangle$$

[Mnemonic : 23, 31, 12.]

This definition seems strange, but the whole point is that the vector $\vec{u} \times \vec{v}$ is simultaneously perpendicular to the vectors $\vec{u}$ & $\vec{v}$:

$$\vec{u} \cdot (\vec{u} \times \vec{v}) = 0$$
$$\vec{v} \cdot (\vec{u} \times \vec{v}) = 0$$

Easy enough to check:

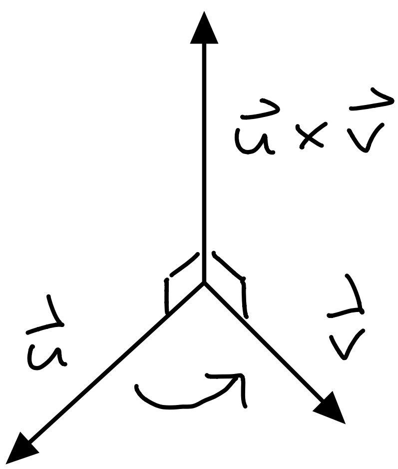
$$\vec{u} \cdot (\vec{u} \times \vec{v})$$

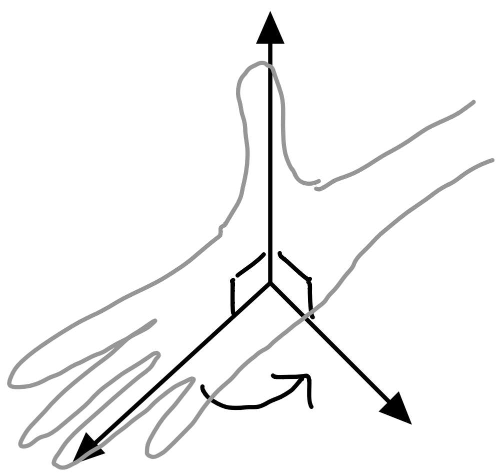
$$= u_1(u_2v_3 - u_3v_2) + u_2(u_3v_1 - u_1v_3) + u_3(u_1v_2 - u_2v_1)$$

$$= \begin{array}{l} \cancel{u_1u_2}v_3 - \cancel{u_1u_3}v_2 \\ + \cancel{u_2u_3}v_1 - \cancel{u_2u_1}v_3 \\ + \cancel{u_3u_1}v_2 - \cancel{u_3u_2}v_1 \end{array} = 0 \checkmark$$

That was lucky!

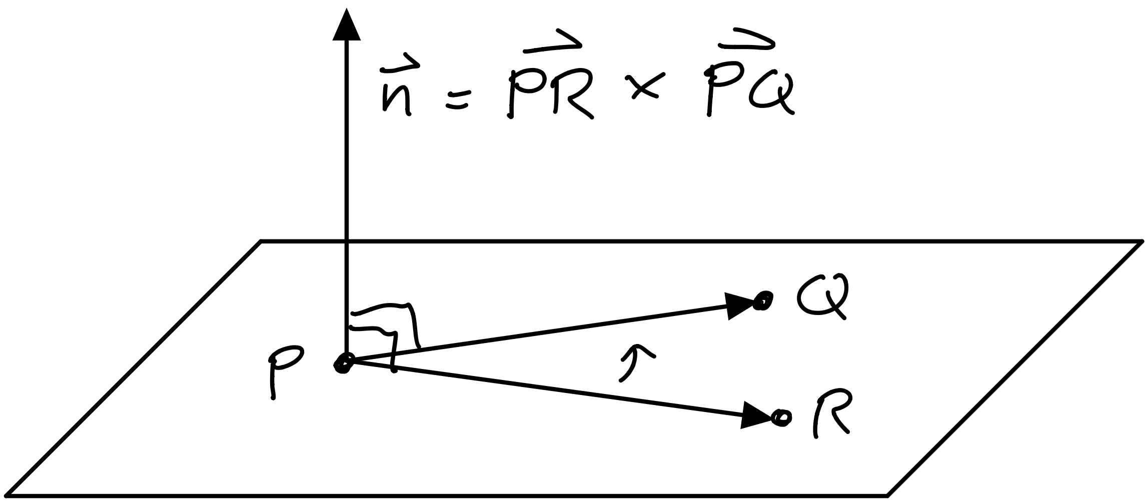
Picture: "Right Hand Rule"





As your fingers sweep from $\vec{u}$ to $\vec{v}$, your thumb points in the direction of $\vec{u} \times \vec{v}$.

We will use this to find a normal vector to our plane:



In our example:

$$\vec{PQ} = \vec{u} = \langle 1, 1, 1 \rangle$$

$$\vec{PR} = \vec{w} = \langle 1, 2, 3 \rangle$$

so let's take

$$\vec{n} = \vec{w} \times \vec{u}$$

$$= \langle 1, 2, 3 \rangle \times \langle 1, 1, 1 \rangle$$

$$= \langle 2\cdot 1 - 3\cdot 1,\ 3\cdot 1 - 1\cdot 1,\ 1\cdot 1 - 2\cdot 1 \rangle$$

$$= \langle -1, 2, -1 \rangle$$

Check :

$$\langle 1,1,1\rangle \cdot \langle -1,2,-1\rangle = -1+2-1 = 0 \checkmark$$

$$\langle 1,2,3\rangle \cdot \langle -1,2,-1\rangle = -2+4-3 = 0 \checkmark$$

Finally, taking

$$(x_0, y_0, z_0) = P = (4,1,2)$$

$$\langle a,b,c\rangle = \langle -1,2,-1\rangle$$

gives the equation of our plane :

$$a(x-x_0)+b(y-y_0)+c(z-z_0) = 0$$

$$-(x-4)+2(y-1)-(z-2) = 0$$

$$-x+2y-z+4-2+2 = 0$$

$$-x+2y-z = -4$$

$$x - 2y + z = 4$$

[See HW 1 Problem 5(c).]

Remark : There were 3 points we could have chosen

$$(x_0, y_0, z_0) = P \text{ or } Q \text{ or } R$$

and 6 possible vectors

$$\vec{n} = \vec{u} \times \vec{v}$$
$$\text{or } \vec{v} \times \vec{u}$$
$$\text{or } \vec{u} \times \vec{w}$$
$$\text{or } \ldots$$

Thus there were 18 possible choices. But it doesn't matter because they all lead to the

SAME EQUATION !