HW 4 due Mon 11:40am

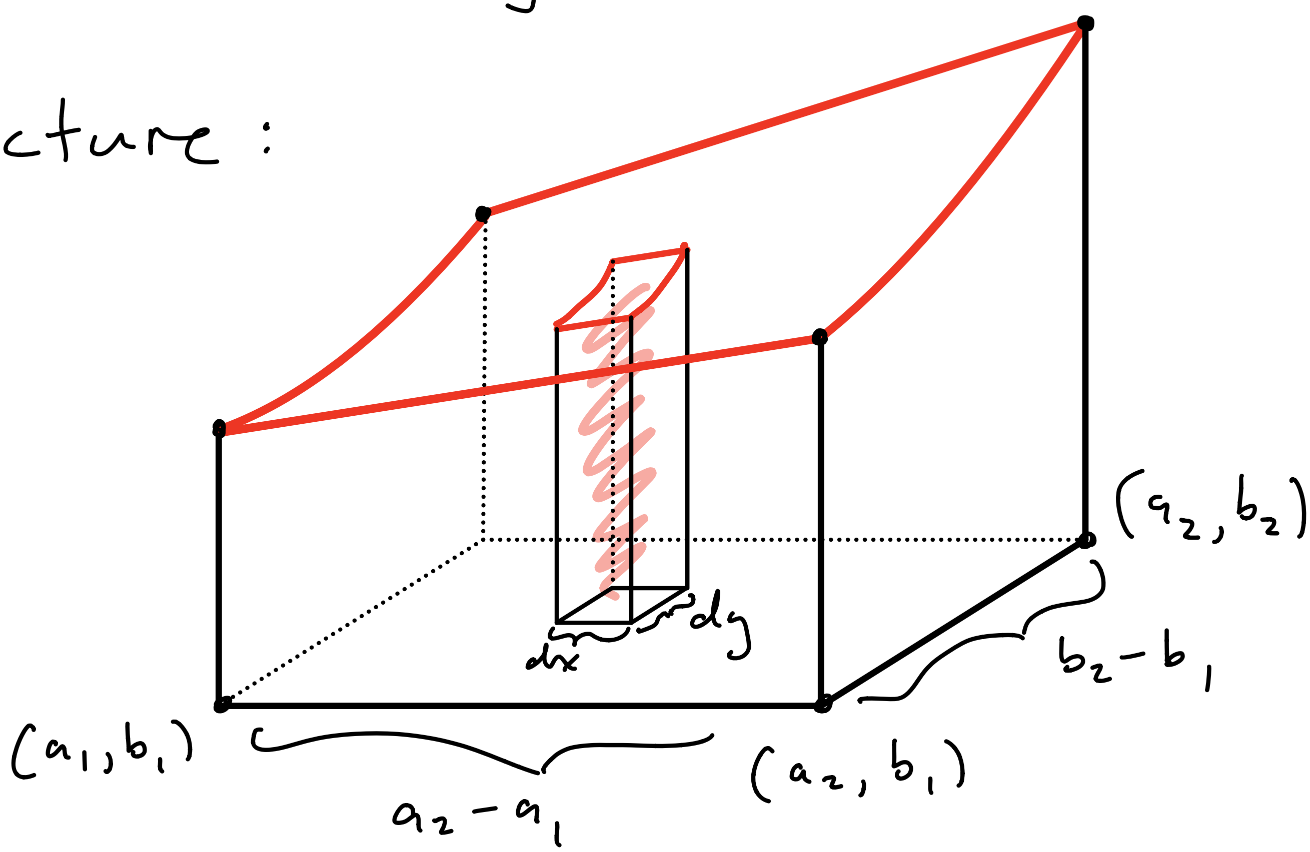
Quiz 4 on Tues at 11:40am

Chapter 5: Multiple Integration

Last time we computed the volume between the surface $z = xy^2$ and a rectangle in the x,y-plane:

$$a_1 \le x \le a_2,$$
$$b_1 \le y \le b_2.$$

Picture:



Above the point (x,y) is an infinitesimally skinny column with

$$\text{volume} = (\text{height})(\text{area of base})$$
$$= xy^2\,dx\,dy$$

To compute the total volume we just "sum up all the columns":

$$\text{vol} = \iint\limits_{\substack{(x,y)\text{ in} \\ \text{the rectangle}}} \begin{matrix}\text{volume of the skinny} \\ \text{column above } (x,y)\end{matrix}$$

[Remark: "$\int$" is a stretched "S".]

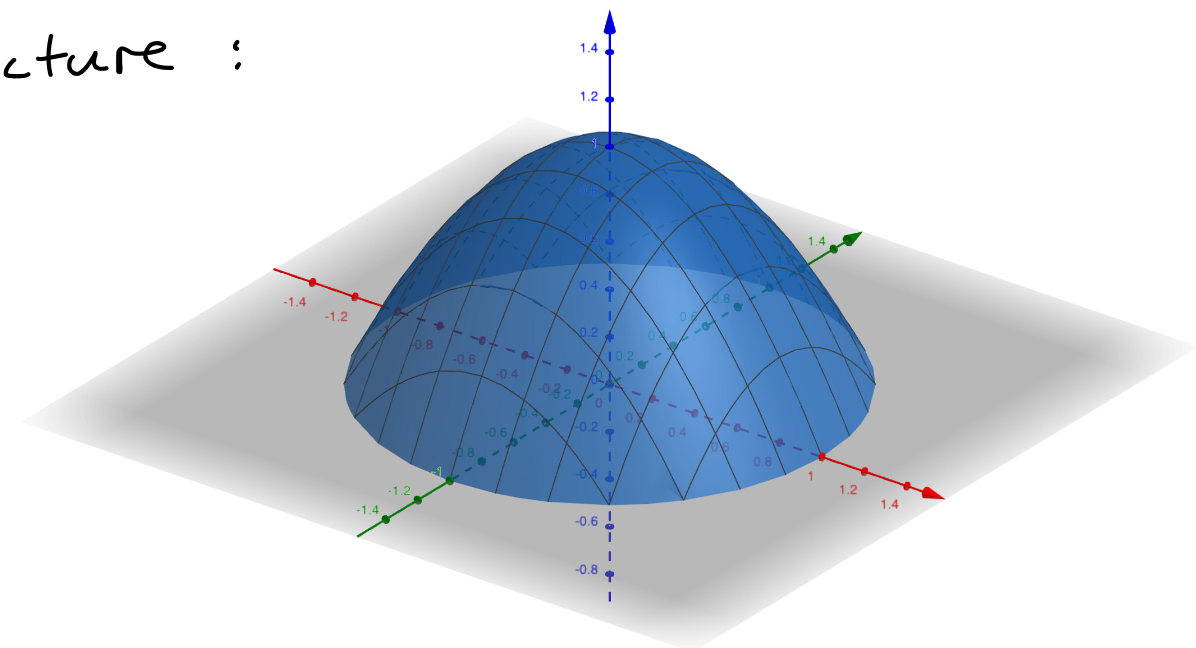
$$\text{vol} = \int_{y=b_1}^{y=b_2} \int_{x=a_1}^{x=a_2} xy^2\,dx\,dy$$

We can compute the two integrals in either order.

[Remark: Volume below the x,y-plane counts as negative.]

A harder example. Let's compute the volume of the "parabolic dome" above the unit disk $x^2+y^2 \le 1$ and below the surface $z = 1-x^2-y^2$.

Picture:



[Think: It fits inside the cylinder of radius 1 & height 1, so we know that vol $< \pi$.]

We will compute this by summing the volumes of many skinny columns:

$$\text{vol} = \iint \begin{matrix}\text{volume of the skinny} \\ \text{column above } (x,y)\end{matrix}$$

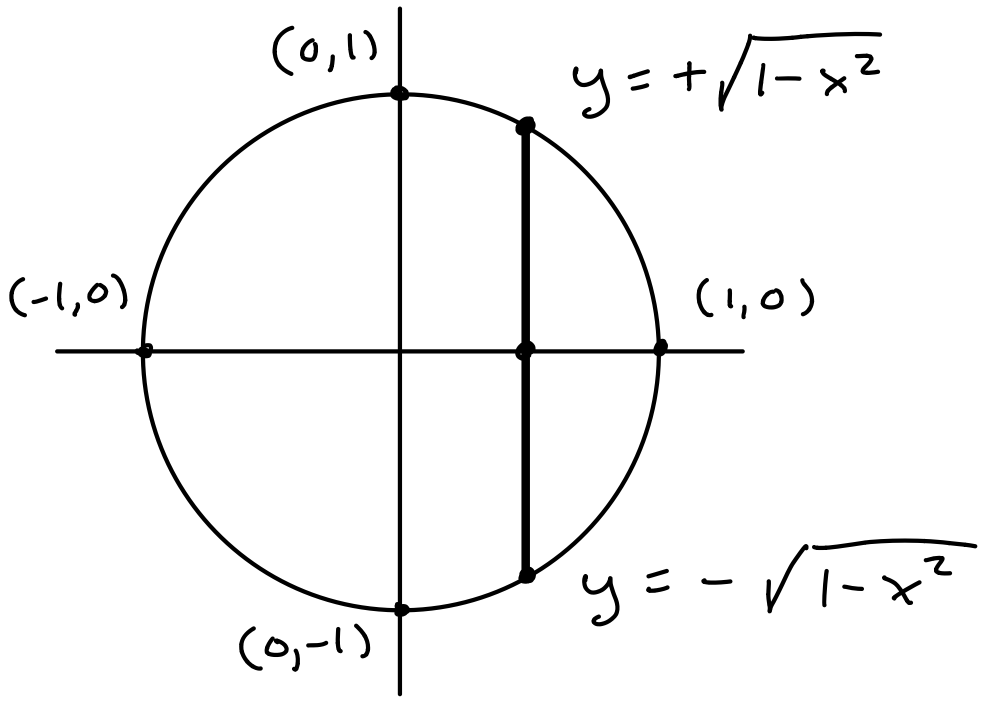
$$= \iint_{\substack{(x,y) \text{ in the} \\ \text{unit disk}}} (1-x^2-y^2)\,dx\,dy$$

This time the difficulty is to find the bounds:

$$? \leq x \leq ?$$
$$? \leq y \leq ?$$

There are two basic ways to do it:

For a fixed value of x we must have $-\sqrt{1-x^2} \leq y \leq +\sqrt{1-x^2}$.



So we can "parametrize the disk" as follows:

$$-1 \leq x \leq +1$$
$$-\sqrt{1-x^2} \leq y \leq +\sqrt{1-x^2}$$

[The other method is to fix y.]

If we use these bounds then we <u>must</u> integrate over y <u>first</u>:

$$\text{vol} = \int_{x=-1}^{x=+1} \left(\int_{y=-\sqrt{1-x^2}}^{y=+\sqrt{1-x^2}} (1-x^2-y^2)\,dy \right) dx$$

First compute the inside integral:

$$\int_{y=-\sqrt{1-x^2}}^{y=+\sqrt{1-x^2}} (1-x^2-y^2)\,dy$$

$$= \left(y - yx^2 - \frac{1}{3}y^3 \right)_{y=-\sqrt{1-x^2}}^{y=+\sqrt{1-x^2}}$$

$$= \left[(1-x^2)y - \frac{1}{3}y^3 \right]_{y=-\sqrt{1-x^2}}^{y=+\sqrt{1-x^2}}$$

$$= 2\left[(1-x^2)\sqrt{1-x^2} - \frac{1}{3}\left(\sqrt{1-x^2}\right)^3 \right]$$

$$= 2\left[(1-x^2)^{3/2} - \frac{1}{3}(1-x^2)^{3/2} \right]$$

$$= \frac{4}{3}(1-x^2)^{3/2}$$

So the volume is

$$\text{vol} = \int_{x=-1}^{x=+1} \frac{4}{3}(1-x^2)^{3/2}\,dx$$

Do you know how to compute this integral? Neither do I. My computer says that

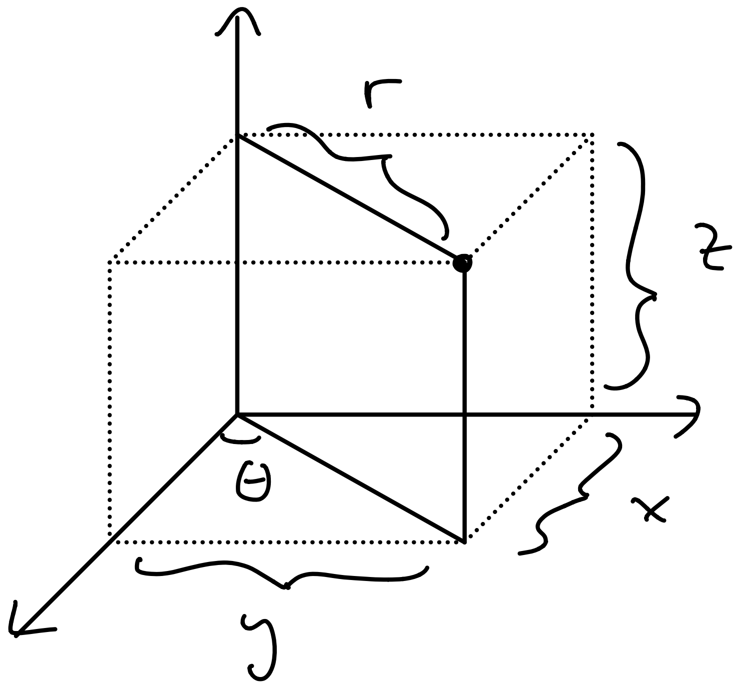
$$\text{vol} = \pi/2 . \qquad (\text{nice})$$

That method worked but it was not easy. Is there a better method? Yes!

Cylindrical coordinates:

Since the parabolic dome has

rotational symmetry around the z-axis, it is better to use so-called "cylindrical coordinates"



We write the Cartesian coords. as

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

Why do we want to do this?

Note that $x^2 + y^2 = r^2$, so the equation of our surface is

$$z = 1 - x^2 - y^2$$

$$z = 1 - r^2$$

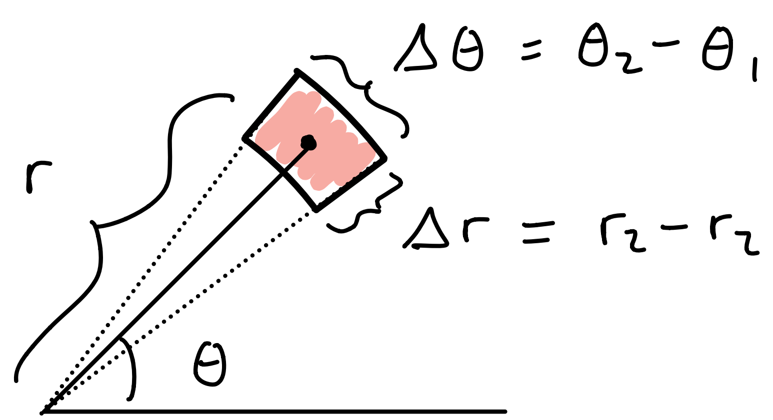
[Nice that it doesn't involve θ. This happens because of the rotational symmetry.]

It is also easy to parametrize the unit circle $x^2 + y^2 \leq 1$ by

$$0 \leq r \leq 1$$

$$0 \leq \theta \leq 2\pi.$$

The only difficulty is to find the area of the base of a skinny column centered at (r, θ):



The area of this region is not

$$\Delta r \, \Delta \theta$$

because it is not a rectangle!

Instead the area is

$$\underbrace{(\pi r_2^2 - \pi r_1^2)}_{\text{area between two circles}} \underbrace{(\theta_2 - \theta_1)/2\pi}_{\text{how much of the circles}}$$

$$= \frac{1}{2}(r_2^2 - r_1^2)(\theta_2 - \theta_1)$$

$$= \frac{1}{2}(r_2 + r_1)(r_2 - r_1)(\theta_2 - \theta_1)$$

$$= \frac{1}{2}(r_2 + r_1)\,\Delta r \, \Delta \theta$$

As $\Delta r \to 0$ & $\Delta \theta \to 0$ then this becomes

$$\frac{1}{2}(r + r)\, dr \, d\theta = r \, dr \, d\theta .$$

Remark: This is a computation that we only have to perform once. From now on we will just MEMORIZE the formula

$$\boxed{dx\,dy = r\,dr\,d\theta}$$

Finally we compute the volume of the parabolic dome:

$$\text{vol} = \iint_{\substack{(r,\theta) \text{ in} \\ \text{unit disk}}} \text{volume of the skinny column centered at } (r,\theta)$$

$$= \iint \underbrace{(1-r^2)}_{\text{height}} \underbrace{r\,dr\,d\theta}_{\text{base}}$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{r=1} (1-r^2)\,r\,dr\,d\theta$$

The integrand doesn't invole θ :)
So we can pull it outside:

$$= \int_{r=0}^{r=1} r(1-r^2)\,dr \int_{\theta=0}^{\theta=2\pi} 1\,d\theta$$

$$= 2\pi \int_{r=0}^{r=1} r(1-r^2)\,dr$$

And this is not so bad!
Let $u = r^2$ so $du = 2r\,dr$. Then

$$\int_{r=0}^{r=1} r(1-r^2)\,dr = \int_{u=0}^{u=1} \frac{1}{2}(1-u)\,du$$

$$= \frac{1}{2}\left(u - \frac{1}{2}u^2\right)\Big|_{u=0}^{u=1}$$

$$= 1/4.$$

And the volume of the dome is

$$\text{vol} = {\color{blue}2\pi} \cdot {\color{red}\frac{1}{4}} = \pi/2 \quad \checkmark$$

The second method was clearly much better. We will use cylindrical coordinates whenever our problem has rotational symmetry around the z-axis.

WARNING: Some problems have no symmetry and integration is generally impossible to do by hand.