HW 4 & Quiz 4 : TBA.

We have covered (the most important sections of) Chapters 1–4. For the rest of the course we will discuss Chapters 5–6.

Chapter 5 : Multiple Integration
Chapter 6 : Vector Calculus

Before we begin Chap 5, here is a preview of Chap 6.

We have discussed 3 kinds of functions in this course:

0) $\mathbb{R} \to \mathbb{R}$ Calc I & II

1) $\mathbb{R} \to \mathbb{R}^n$ parametrized curves

2) $\mathbb{R}^n \to \mathbb{R}$ scalar fields

3) $\mathbb{R}^n \to \mathbb{R}^n$ vector fields

We have studied some kinds of derivatives & integrals, but some are still missing:

	derivatives	integrals
1)	✓	✓
2)	✓	?
3)	?	?

2) Scalar fields can be differentiated in only one way: by the gradient vector. Scalar fields can be integrated over 1D paths, or 2D surfaces, or 3D solid regions, etc.

To make sense of this we will need to discuss multiple integrals (which is Chapter 5).

3) Vector fields can be differentiated in several ways. Let $\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field in 3D space.

We will study the "divergence" $\nabla \cdot \vec{F} : \mathbb{R}^3 \to \mathbb{R}$ and the "curl" $\nabla \times \vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$, defined as follows.

The vector field $\vec{F}$ has 3 components

$$\vec{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle$$

where we can think of the components $P, Q, R : \mathbb{R}^3 \to \mathbb{R}$ as scalar fields.

Definition: The divergence is

$$\text{"} \nabla \cdot \vec{F} \text{"} = \underbrace{\left\langle \frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz} \right\rangle}_{} \cdot \langle P, Q, R \rangle$$

not really a vector.
we call it a "differential operator"

$$= \frac{dP}{dx} + \frac{dQ}{dy} + \frac{dR}{dz}$$

And the curl is

"$\nabla \times \vec{F}$"

$$= \left\langle \frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz} \right\rangle \times \langle P, Q, R \rangle$$

$$= \left\langle \frac{dR}{dy} - \frac{dQ}{dz}, \frac{dP}{dz} - \frac{dR}{dx}, \frac{dQ}{dx} - \frac{dP}{dy} \right\rangle$$

These definitions come from physics; we'll discuss what they mean later.

The highlight of the whole subject are the "Fundamental theorems of vector calculus".

Recall from Calc I:

"Fundamental Theorem of Calculus"

- $$\frac{d}{dx}\int^{x} f(t)\,dt = f(x)$$

- $$\int_a^b f'(t)\,dt = f(b) - f(a)$$

Differentiation & integration are "inverse operations". The same general idea should hold for any kind of differentiation & integration.

- $$\int_{\text{path}} \nabla f = f(\text{tail}) - f(\text{head})$$

  endpoints of the path

- Green / Stokes Theorem:

$$\iint_{\text{surface}} \nabla \times \vec{F} = \int_{\substack{\text{boundary curve}\\ \text{of the surface}}} \vec{F}$$

- Gauss / Divergence Theorem:

$$\underbrace{\iiint}_{\text{solid region}} \nabla \cdot \vec{F} = \underbrace{\iint}_{\text{boundary surface}} \vec{F}$$

And these are all specific examples of a very fancy theorem

$$\text{``} \int_M d\alpha = \int_{\partial M} \alpha \text{''}$$

Never mind. It's not in this course.

You may have noticed the double integral "$\iint$" and the triple integral "$\iiint$". These are the subject of Chapter 5.

Introductory Example:

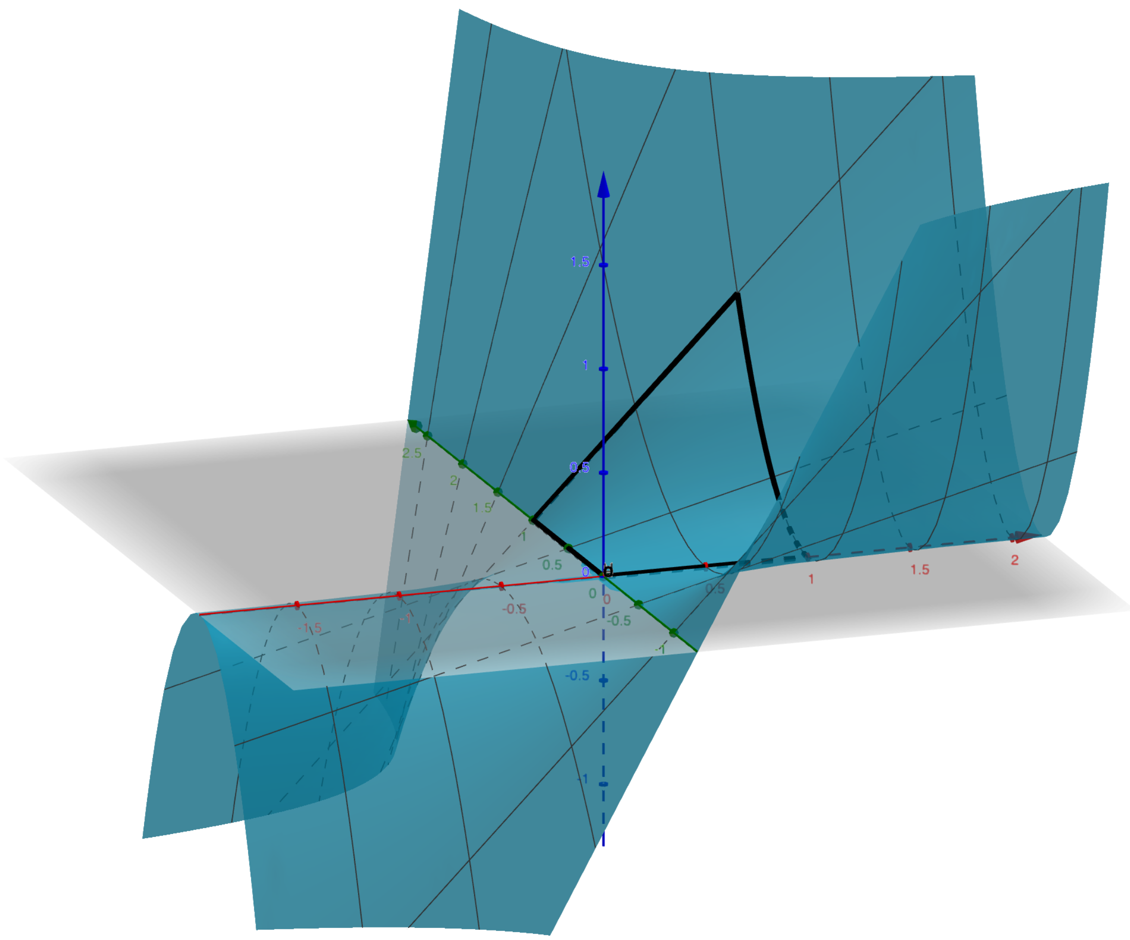
Compute the volume of the solid region above the square

$$0 \le x \le 1$$
$$0 \le y \le 1$$

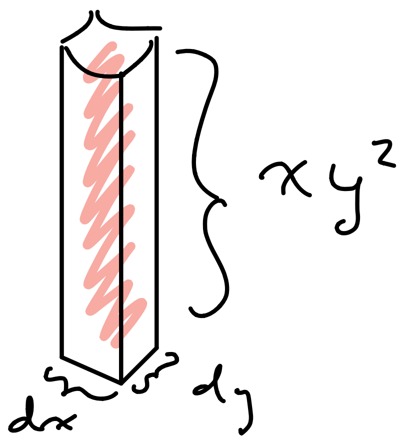
and below the surface

$$z = xy^2 .$$

Picture:



Idea: Above the point (x,y) there is an infinitesimal vertical column of height xy^2 & base area $dx\,dy$.



The volume of the column is

$(\text{height})(\text{area of base}) = xy^2\,dx\,dy$

To get the volume of the whole solid region we just "add up all the little columns":

$$\text{volume} = \iint xy^2\,dx\,dy$$

how to compute?

The notation indicates that we should compute two integrals. The order doesn't matter, so let's integrate over x first:

$$\text{volume} = \int_0^1 \left(\int_0^1 xy^2 \, dx \right) dy$$

$$= \int_0^1 \left(\frac{1}{2} x^2 y^2 \Big|_{x=0}^{x=1} \right) dy$$

$$= \int_0^1 \left(\frac{1}{2} y^2 - 0 \right) dy$$

$$= \left(\frac{1}{6} y^3 \right)_{y=0}^{y=1}$$

$$= \frac{1}{6} - 0 = \frac{1}{6}$$

The exact volume is $1/6$.

Just to check that order doesn't matter, let's do y first:

$$\text{vol} = \int_0^1 \left(\int_0^1 xy^2 \, dy \right) dx$$

$$= \int_0^1 \left(\frac{1}{3} x y^3 \right)_{y=0}^{y=1} dx$$

$$= \int_0^1 \left(\frac{1}{3} x - 0 \right) dx$$

$$= \left(\frac{1}{6} x^2 \right)_{x=0}^{x=1} = \frac{1}{6} \quad \checkmark$$

The same method can be used to compute the volume above any rectangle in the x,y-plane:

$$a_1 \le x \le a_2 \, ,$$

$$b_1 \le y \le b_2 \, .$$

$$\text{vol} = \int_{b_1}^{b_2} \left(\int_{a_1}^{a_2} x y^2 \, dx \right) dy$$

$$= \int_{b_1}^{b_2} \left(\frac{1}{2} (a_2^2 - a_1^2) y^2 \right) dy$$

$$= \frac{1}{6} (a_2^2 - a_1^2)(b_2^3 - b_1^3)$$

Issue : Sometimes this "volume" is zero or negative.

Solution : Volume <u>below</u> the x,y-plane will count as negative. So the integral is really a "signed volume"

Example :

$$\int_{0}^{1} \left(\int_{-1}^{1} x y^2 \, dx \right) dy = 0$$

because the positive & negative volumes exactly cancel.

Picture: