Slight schedule delay:

- HW 3 due Monday
- Quiz 3 on Tuesday

Chapter 4: Theory of the Gradient.

Recall: Let $f(x_1, x_2, \ldots, x_n)$ be a scalar-valued function of n variables, i.e.,

$$f : \mathbb{R}^n \to \mathbb{R}$$

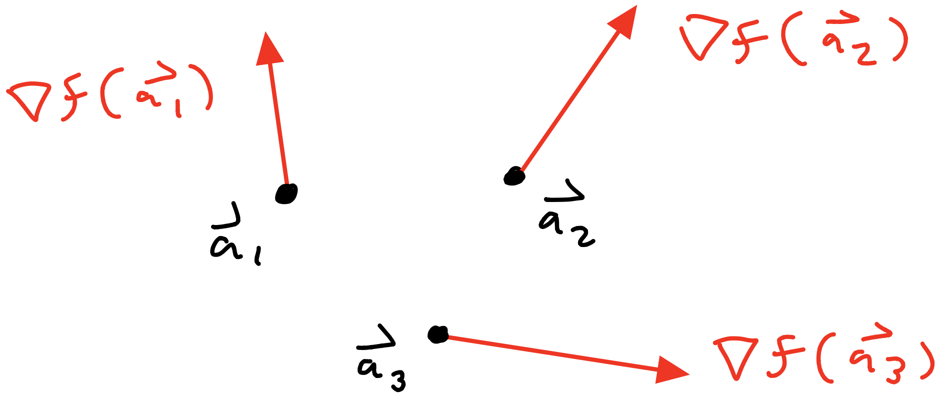
Then at each point $\vec{a} = (a_1, a_2, \ldots, a_n)$ we have a gradient vector:

$$\nabla f = \left\langle \frac{df}{dx_1}, \frac{df}{dx_2}, \ldots, \frac{df}{dx_n} \right\rangle$$

Then the vector $\nabla f(\vec{a})$ is obtained by evaluating each derivative at the point $\vec{a}$:

$$\nabla f(\vec{a}) = \left\langle \frac{df}{dx_1}(\vec{a}), \ldots, \frac{df}{dx_n}(\vec{a}) \right\rangle$$

Mental Picture :



At each point in space we have a vector. Thus we can also think of " ∇f " as a function

$$\nabla f : \mathbb{R}^n \to \mathbb{R}^n$$

$$\text{point} \longmapsto \text{vector at this point}$$

Sometimes we call this a

"vector field".

Summary : The "nabla operator ∇" turns each "scalar field" $f: \mathbb{R}^n \to \mathbb{R}$ into a "vector field" $\nabla f : \mathbb{R}^n \to \mathbb{R}^n$.

Example: Let $f(x,y)$ be the height of a mountain above the point (x,y) in the x,y-plane. Say that

$$f(x,y) = 100 - 2x^2 - y^2.$$

In this case, ∇ tells us the local direction that points "directly uphill":

$$\nabla f(x,y) = \left\langle \frac{df}{dx}, \frac{df}{dy} \right\rangle$$

$$= \langle 0 - 4x - 0, 0 - 0 - 2y \rangle$$

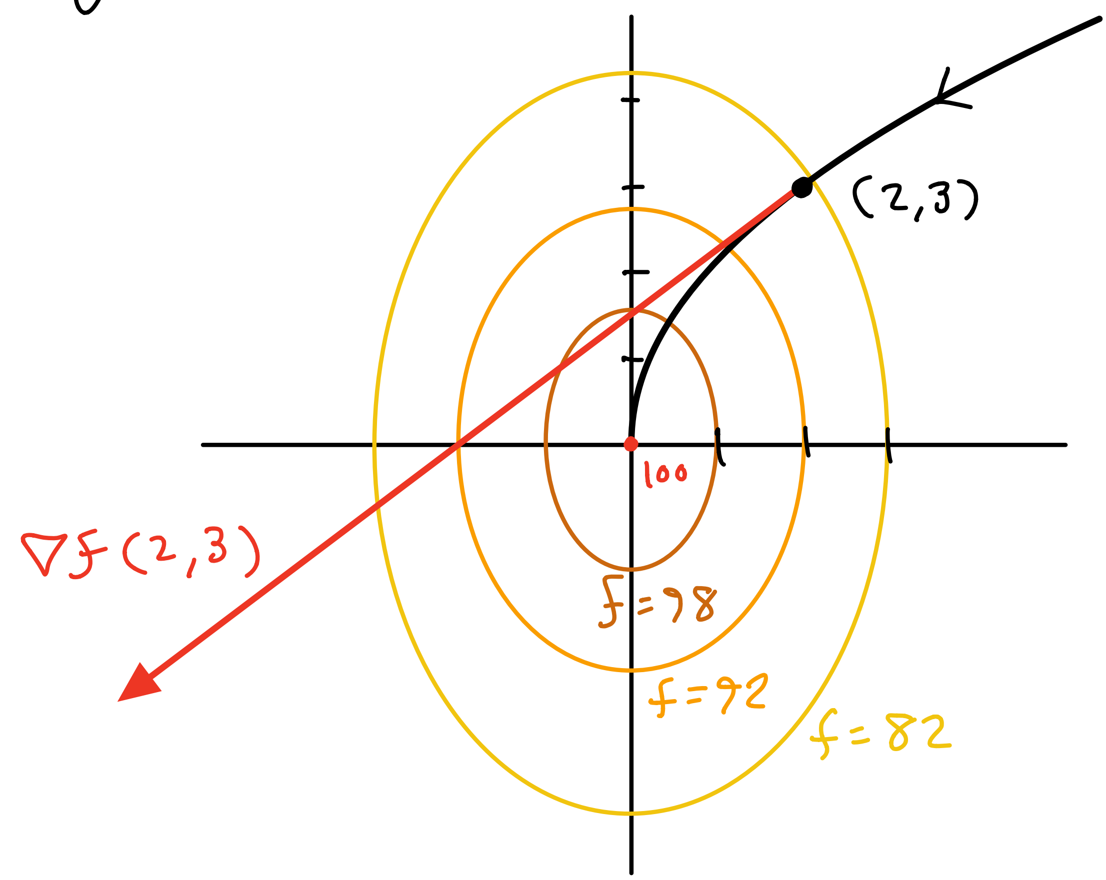
$$= \langle -4x, -2y \rangle$$

So, e.g., if you are at the point $(2,3)$ then you should travel in the horizontal direction

$$\nabla f(2,3) = \langle -4 \cdot 2, -2 \cdot 3 \rangle = \langle -8, -6 \rangle$$

to go "directly uphill". But this is only correct near the point (2,3).

To keep going "directly uphill" you will need to change your direction as you move. The resulting path is called the "gradient flow":



Later we will see that the curve of gradient flow is actually a parabola.

[Idea: We don't head straight to

the top of the hill because we don't know where it is. Instead we just pick the steepest direction near to us. Eventually we <u>hope</u> to reach the top of the hill!]

Example: Consider the function

$$f(x,y,z) = 5x^2 - 3xy + xyz$$

Let's compute the derivatives:

- $df/dx = 10x - 3y + yz$

  [treat y & z like constants]

- $df/dy = 0 - 3x + xz$

  [treat x & z as constants]

- $df/dz = 0 - 0 + xy$

  [treat x & y as constants]

$$\nabla f = \langle 10x - 3y + yz, -3x + xz, xy \rangle$$

Summary: At each point $\vec{a} = (a, b, c)$ in $\mathbb{R}^3$ we have a vector

$$\nabla f(a,b,c) = \langle 10a - 3b + bc, -3a + ac, ab \rangle$$

Unlike the previous example, we cannot view $f(x,y,z)$ as the height of a hill [the hill would live in 4D space!] but we could view f as a temperature distribution. Then ∇f tells you the best local direction to increase your temperature.

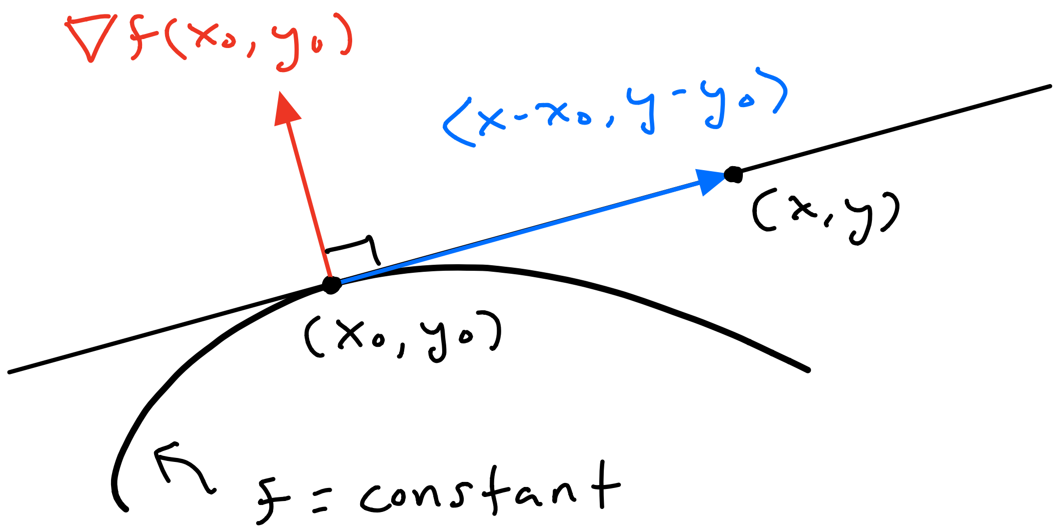
That's the interpretation of ∇ in physics. The interpretation in geometry has to do with tangent lines & planes.

We start with the case

$f : \mathbb{R}^2 \to \mathbb{R}$

For any constant c, the equation $f(x,y) = c$ defines a <u>curve</u> in $\mathbb{R}^2$.

For any point (x_0, y_0) on this curve [i.e. such that $f(x_0,y_0) = c$], the tangent line at this point is $\perp$ to the gradient vector $\nabla f(x_0,y_0)$:



Thus we obtain an equation for the tangent line through (x_0,y_0):

perpendicular!

$$\nabla f(x_0,y_0) \cdot \langle x - x_0, y - y_0 \rangle = 0$$

$$\left\langle \frac{df}{dx}(x_0,y_0), \frac{df}{dy}(x_0,y_0) \right\rangle \cdot \langle x-x_0, y-y_0 \rangle = 0$$

$$\frac{df}{dx}(x_0,y_0)(x-x_0) + \frac{df}{dy}(x_0,y_0)(y-y_0) = 0$$

That's a lot of symbols. To clean it up we can write $f_x = df/dx$ & $f_y = df/dy$ to get

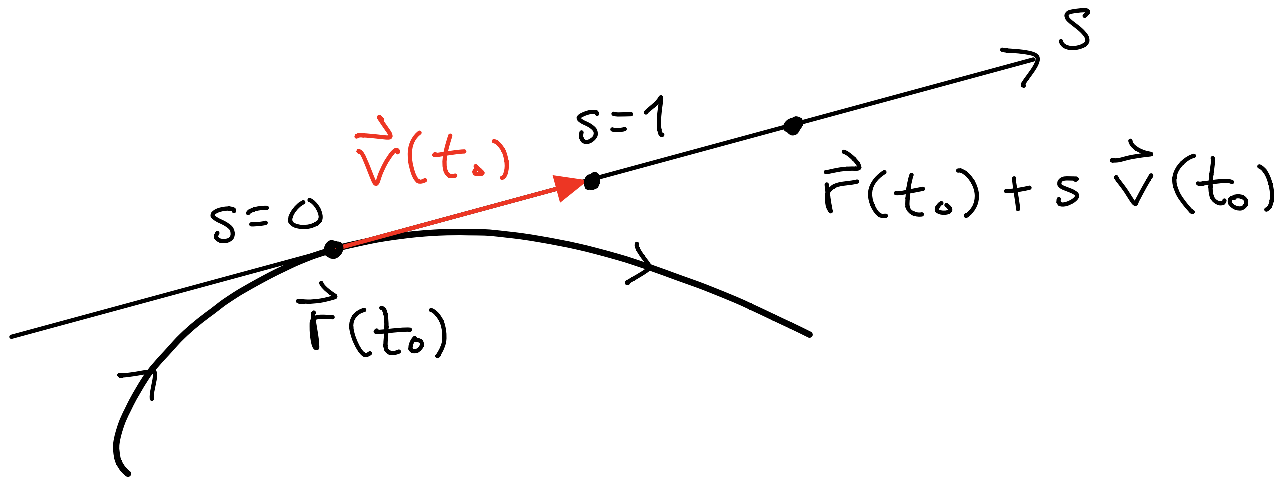
$$f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) = 0.$$

Sometimes people even write

$$f_x = f_x(x_0,y_0),$$

but that becomes dangerous!

Compare that to the old method where we have a parametrization of the curve $\vec{r}(t)$. Then the tangent line at time t_0 has the form $\vec{r}(t_0) + s\,\vec{v}(t_0)$:



Example: Find the tangent line to the curve $xy = 1$ at the point $(x_0, y_0) = (2, 1/2)$.

- use the gradient vector.

Define $f(x,y) = xy$, so the curve is $f(x,y) = 1$. The gradient is

$$\nabla f(x,y) = \langle f_x, f_y \rangle = \langle y, x \rangle$$

$$\nabla f(2, 1/2) = \langle 1/2, 2 \rangle$$

The tangent line is

$$\langle 1/2, 2 \rangle \cdot \langle x-2, y-1/2 \rangle = 0$$

$$1/2(x-2) + 2(y-1/2) = 0$$

$$\frac{1}{2}x - 1 + 2y - 1 = 0$$

$$2y = -\frac{1}{2}x + 2$$

$$y = -\frac{1}{4}x + 1$$

- Use a paratrization, say

$$\vec{r}(t) = (t, 1/t),$$

$$\vec{v}(t) = \langle 1, -1/t^2 \rangle.$$

The velocity vector at $\vec{r}(2) = (2, 1/2)$ is $\vec{v}(2) = \langle 1, -1/4 \rangle$, so the parametrized tangent line is

$$\langle x(s), y(s) \rangle = \vec{r}(2) + s\,\vec{v}(2)$$

$$= \langle 2, 1/2 \rangle + s \langle 1, -1/4 \rangle$$

$$= \langle 2 + s, \frac{1}{2} - \frac{1}{4}s \rangle$$

We can elimininate the parameter:

$$x = 2 + s \qquad \qquad s = x - 2$$

$$y = \frac{1}{2} - \frac{1}{4}s \longrightarrow s = (y - \frac{1}{2}) / -\frac{1}{4}$$

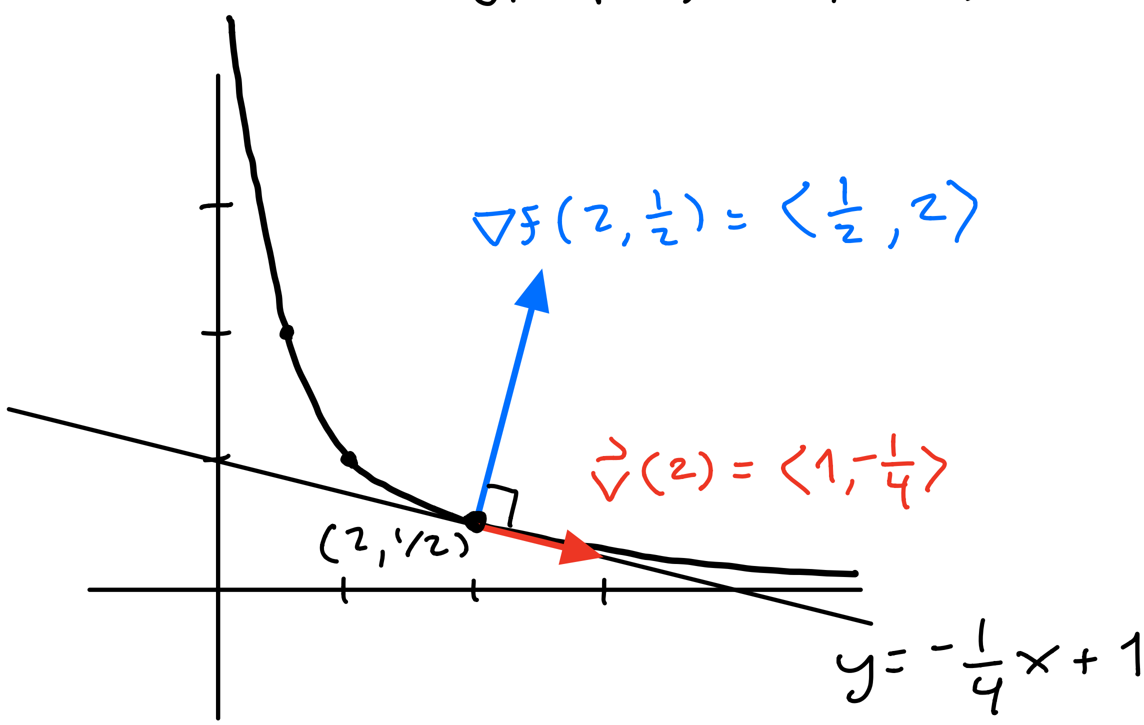
to get

$$\frac{y - 1/2}{-1/4} = x - 2$$

$$y - 1/2 = -\frac{1}{4}(x - 2)$$

$$y = -\frac{1}{4}x + 1 \quad \checkmark$$

Picture : The curve is $f(x,y) = xy = 1$
or $\vec{r}(t) = (t, 1/t)$.



- There is even one more method! In Calc I you used a sort of fake method called "implicit differentiation".

The slope of a curve is dy/dx.
Apply the "d/dx operator" to both sides of the equation:

$$xy = 1$$

$$\frac{d}{dx}(xy) = \frac{d}{dx}1$$

$$\overset{1}{\cancel{\frac{dx}{dx}}}\, y + x\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -y/x$$

At the point $(x,y) = (2, 1/2)$ this gives slope $= dy/dx = -1/2 / 2 = -1/4$ as expected. ✓

What happens in $\mathbb{R}^3$?

Consider the function

$$f(x,y,z) = (2x)^2 + y^2 + z^2.$$

The "level surfaces" of this "scalar field" are ellipsoids. Consider

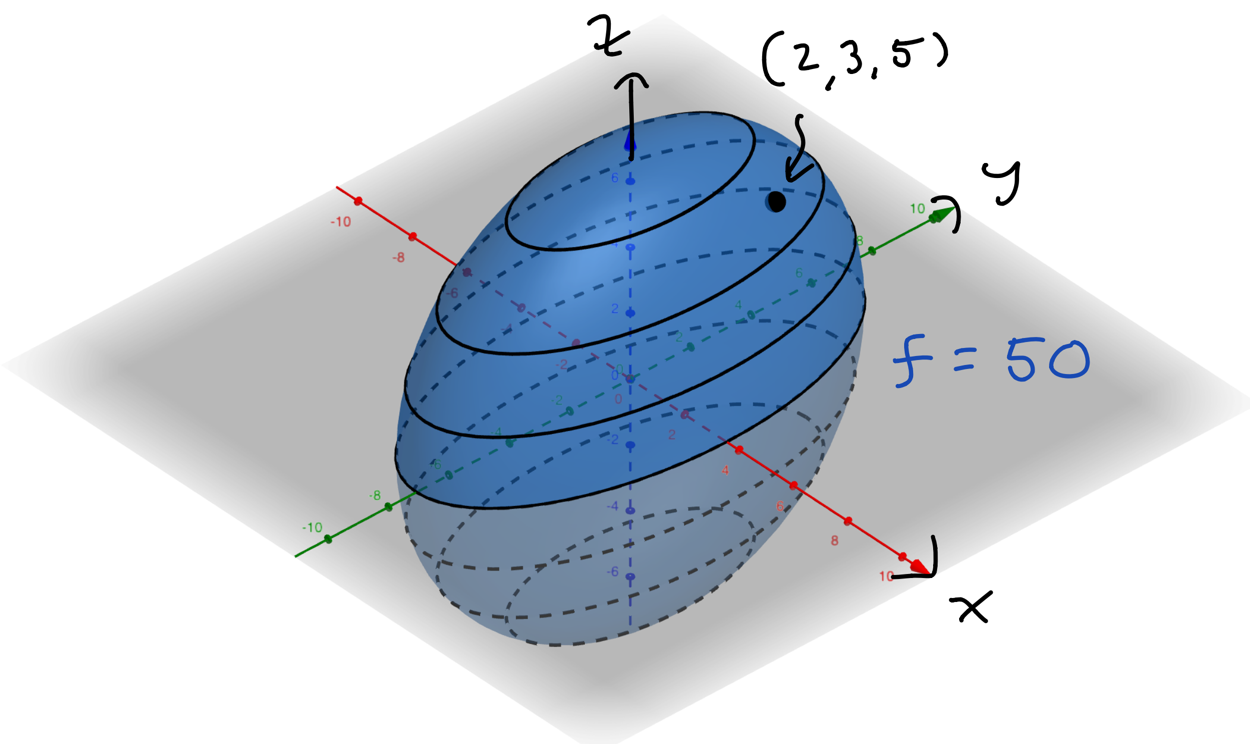
$$f(x,y,z) = 50$$

$$(2x)^2 + y^2 + z^2 = 50$$

We note that the point $(2,3,5)$ is on this surface:

$$f(2,3,5) = (4)^2 + 3^2 + 5^2 = 50.$$

Here is a picture:



We want to find the tangent plane to this ellipsoid at this point.

It turns out that the tangent plane is orthogonal to the gradient vector

$$\nabla f = \langle 8x, 2y, 2z \rangle$$

$$\nabla f(2,3,5) = \langle 16, 6, 10 \rangle,$$

so the equation of the tangent plane is

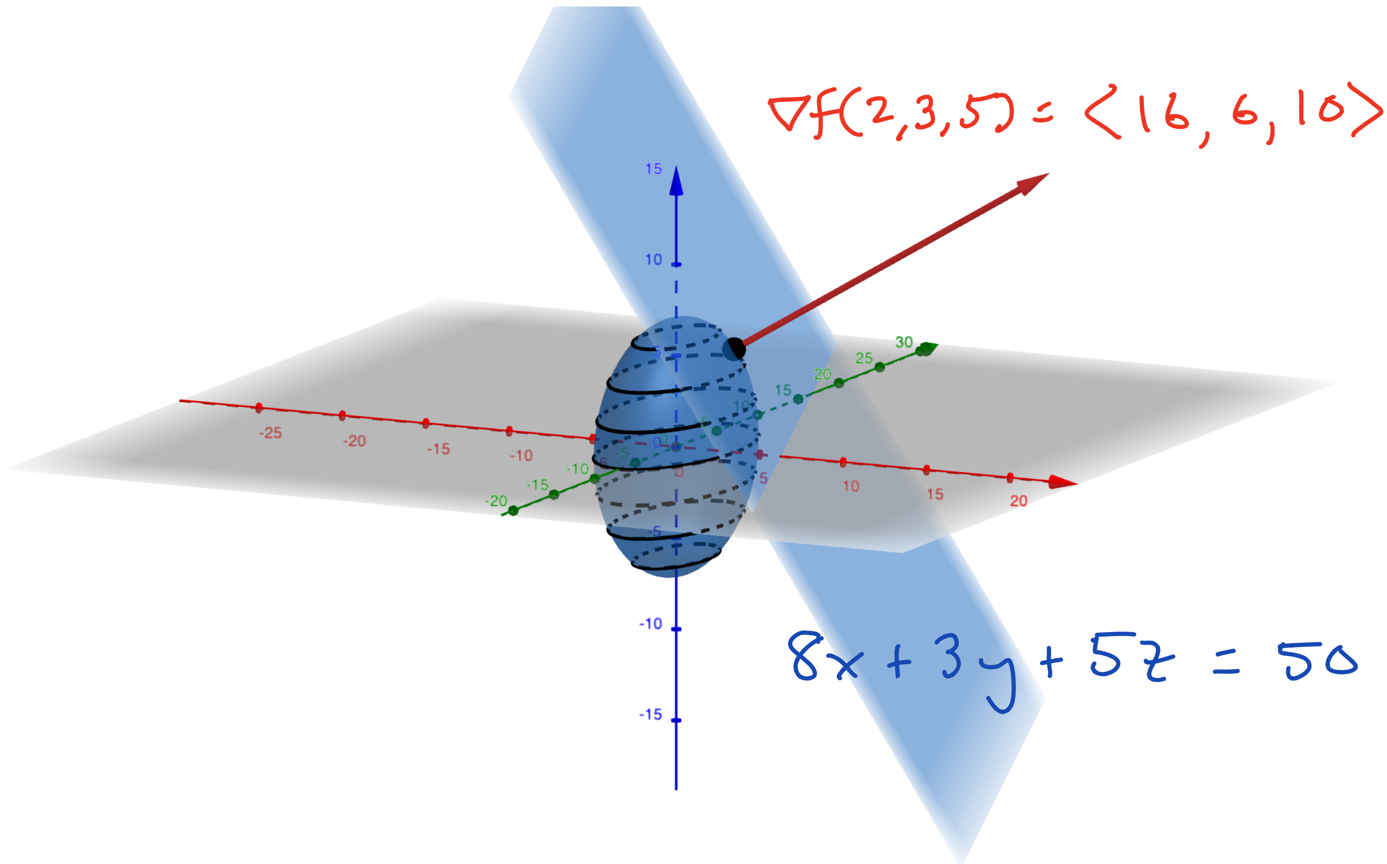
$$16(x-2) + 6(y-3) + 10(z-5) = 0$$

$$16x + 6y + 10z = 32 + 18 + 50$$

$$16x + 6y + 10z = 100$$

$$8x + 3y + 5z = 50$$

We add this information to our picture:



Summary: The tangent plane to the surface $f(x,y,z) = \text{constant}$ at the point (x_0, y_0, z_0) has equation

$$\nabla f(x_0, y_0, z_0) \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$