HW 5 due tomorrow.

Quiz 5 Wed.

Final Proj due Friday.

Last time we discussed the most basic form of Green's Theorem.

Today I'll present a more general form.

Green's Theorem:

Let D be a 2D region with "boundary curve" ∂D. If a vector field $\vec{F} = \langle P, Q \rangle$ is defined at every point of D, then

$$\iint_D \operatorname{curl}(\vec{F})\, dA = \oint_{\partial D} \vec{F} \cdot \vec{T}\, ds$$

$$\iint_D (Q_x - P_y)\,dx\,dy$$

$$= \oint_{\partial D} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$$

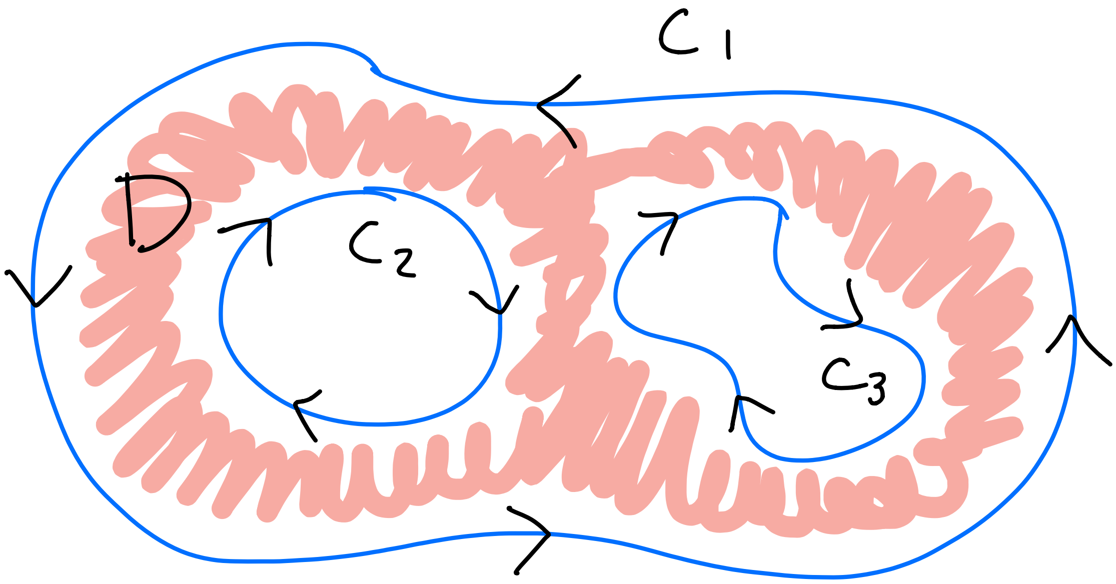
$$= \oint_{\partial D} \langle P, Q \rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle dt$$

$$= \oint_{\partial D} P\,dx + Q\,dy$$

These are just different notations for the same idea:

integral of curl($\vec{F}$) over D = circulation of $\vec{F}$ along ∂D

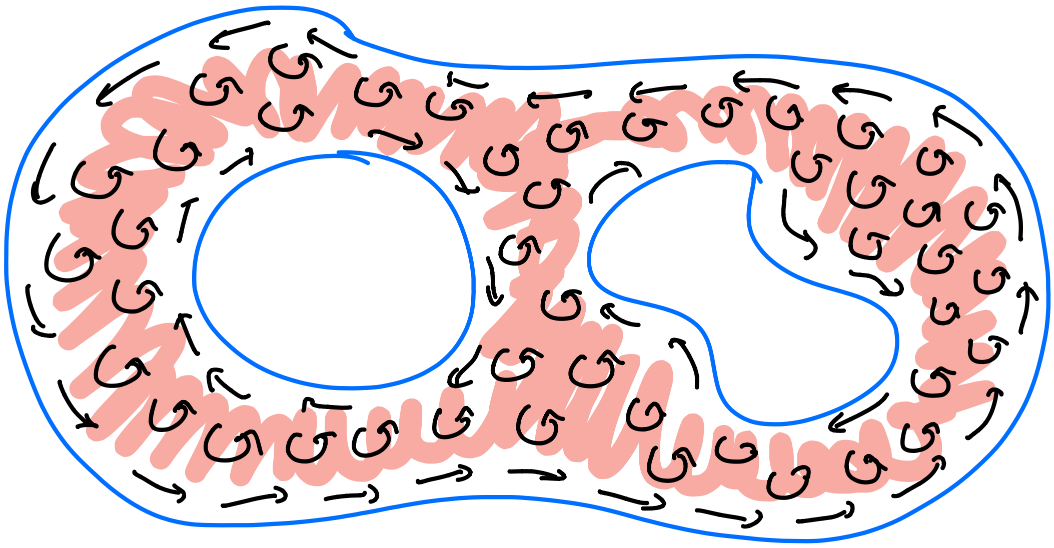
What makes this form more general is that the "boundary curve" ∂D is allowed to have multiple pieces:



In this picture, the boundary is the "sum" of three curves:

$$\partial D = C_1 + C_2 + C_3$$

The only rule is that the curves are oriented so that the region D is always "to the left". Then the idea of the proof is the same as before:



The rotations in the interior cancel, leaving only the circulation along the boundary.

Example: Consider the vector field

$$\vec{F} = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$$

We showed last time that

$$\operatorname{curl}(\vec{F})(x,y) = \begin{cases} 0 \text{ if } (x,y) \neq (0,0) \\ \text{undefined if } (x,y) = (0,0) \end{cases}$$

If C is a simple, connected, counterclockwise loop then I claim

$$\oint_C \vec{F} \cdot \vec{T}\, ds = \begin{cases} 2\pi & \text{if } C \text{ contains } (0,0) \\ 0 & \text{if } C \text{ does not contain } (0,0) \end{cases}$$

Proof: If C does not contain $(0,0)$ then $\operatorname{curl}(\vec{F}) = 0$ at every point inside the loop, so Green's Theorem says

$$\oint_C \vec{F} \cdot \vec{T}\, ds = \iint_{\text{inside}} 0\, dA = 0.$$

For the other statement, let C_1 & C_2 be any two loops containing $(0,0)$.

Here is a picture, assuming that the two loops do not intersect:

If D is the region between the curves then we must have

$$\partial D = C_1 - C_2$$

we need to reverse the orientation of C_2 so that D is always "to the left" of ∂D

Then since $\text{curl}(\vec{F}) = 0$ at every point of D, Green's Theorem says

$$0 = \iint_D \text{curl}(\vec{F})\, dA$$

$$= \int_{C_1 - C_2} \vec{F} \cdot \vec{T} \, ds$$

$$= \int_{C_1} \vec{F} \cdot \vec{T} \, ds - \int_{C_2} \vec{F} \cdot \vec{T} \, ds$$

and hence

$$\int_{C_1} \vec{F} \cdot \vec{T} \, ds = \int_{C_2} \vec{F} \cdot \vec{T} \, ds$$

[If the curves C_1 & C_2 intersect then this is still true but the picture is more complicated.]

Thus we only need to compute the circulation around one specific curve that contains $(0,0)$. The easiest choice is the unit circle C:

$$\vec{r}(t) = \langle \cos t, \sin t \rangle,$$

$$\vec{r}\,'(t) = \langle -\sin t, \cos t \rangle,$$

$$\vec{F}(\vec{r}(t)) = \frac{1}{\cos^2 t + \sin^2 t} \langle -\sin t, \cos t \rangle$$

$$= \langle -\sin t, \cos t \rangle,$$

so that

$$\oint_C \vec{F} \cdot \vec{T}\, ds = \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

$$= \int_0^{2\pi} \langle -\sin t, \cos t \rangle \cdot \langle -\sin t, \cos t \rangle\, dt$$

$$= \int_0^{2\pi} (\sin^2 t + \cos^2 t)\, dt$$

$$= \int_0^{2\pi} 1\, dt = 2\pi \quad \checkmark$$

That's pretty amazing! Today we will see what this result has to do with gravity & electromagnetism.

The "Flux Form" of Green's Theorem:

Given a vector field $\vec{F} = \langle P, Q \rangle$, we may consider the vector field

$$\vec{G} = \langle U, V \rangle = \langle -Q, P \rangle$$

[We rotated $\vec{F}$ by $90°$.] Let's apply Green's Theorem to $\vec{G}$:

$$\iint_D (V_x - U_y)\, dx\, dy = \oint_{\partial D} U\, dx + V\, dy$$

$$\iint_D (P_x + Q_y)\, dx\, dy = \oint_{\partial D} -Q\, dx + P\, dy$$

$$\iint_D \nabla \cdot \langle P, Q \rangle\, dx\, dy = \int_{\partial D} \langle P, Q \rangle \cdot \langle dy, -dx \rangle$$

" $\iint_D$ divergence = Flux across ∂D "

WHAT ?

Given a vector field $\vec{F} : \mathbb{R}^n \to \mathbb{R}^n$

we define a scalar field $\nabla \cdot \vec{F} : \mathbb{R}^n \to \mathbb{R}$

called the "divergence of $\vec{F}$":

$$\nabla \cdot \vec{F} = \text{“}\langle \partial_{x_1}, \dots, \partial_{x_n} \rangle \cdot \langle P_1, \dots, P_n \rangle\text{”}$$

$$= \underbrace{\frac{dP_1}{dx_1} + \frac{dP_2}{dx_2} + \cdots + \frac{dP_n}{dx_n}}_{\text{this is a scalar field}}$$

Sometimes we also write

$$\nabla \cdot \vec{F} = \text{“}\operatorname{div}(\vec{F})\text{”}$$

Special Cases:

- $$\vec{F} = \langle P, Q, R \rangle$$
  $$\nabla \cdot \vec{F} = P_x + Q_y + R_z$$

- $$\vec{F} = \langle P, Q \rangle$$
  $$\nabla \cdot \vec{F} = P_x + Q_y$$

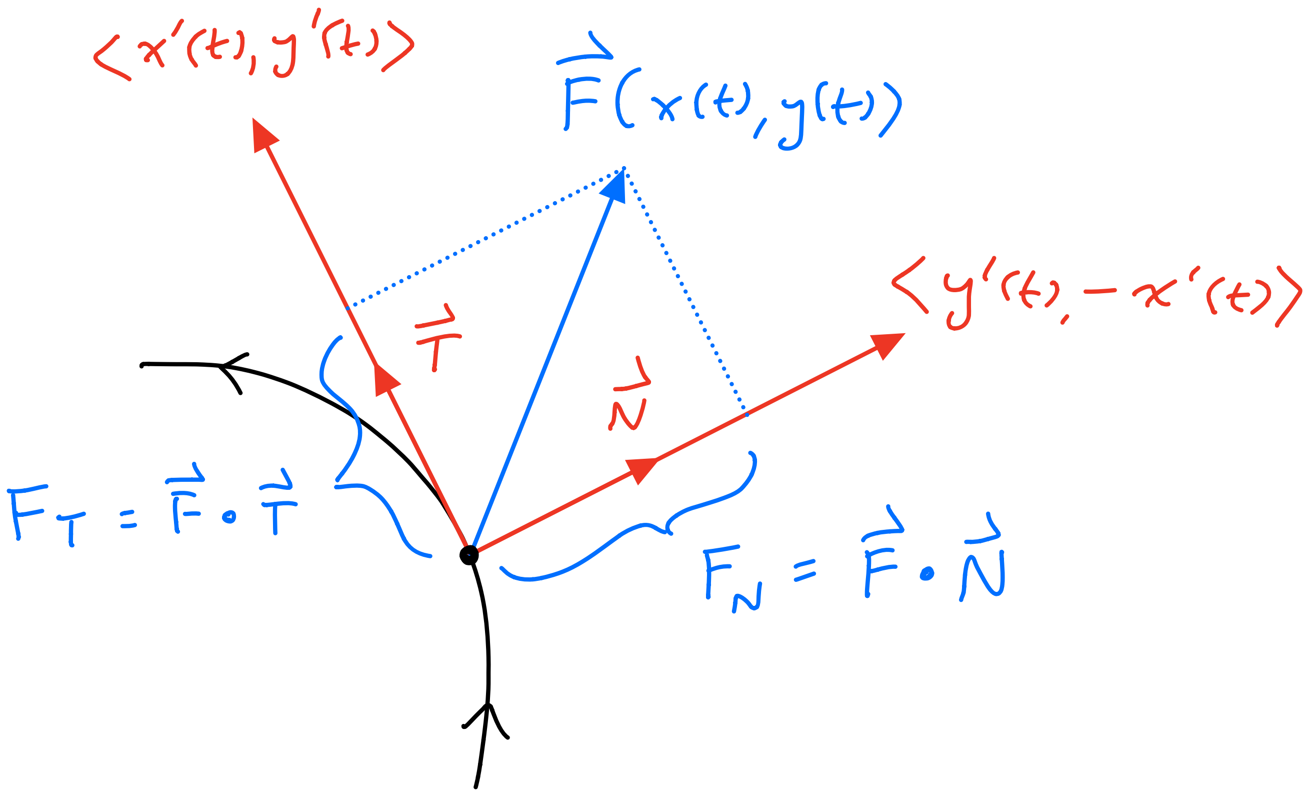
The flux form of Green's Theorem involves the divergence:

$$\iint_D \operatorname{div}(\vec{F})\, dA = \int_{\partial D} \langle P, Q \rangle \cdot \langle dy, -dx \rangle$$

$$= \int_{\partial D} \vec{F}(\vec{r}(t)) \cdot \left\langle \frac{dy}{dt}, -\frac{dx}{dt} \right\rangle dt$$

This is called the "Flux of $\vec{F}$ across the curve ∂D"

Picture :



Given a parametrized curve $\vec{r}(t) = \langle x(t), y(t)\rangle$ we have a velocity $\vec{r}\,'(t) = \langle x'(t), y'(t)\rangle$ & a unit tangent vector

$$\vec{T}(\vec{r}(t)) = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}.$$

To compute the circulation of a

vector field $\vec{F}$ along $\vec{r}(t)$ we observe that the component of $\vec{F}$ in the tangent direction is :

$$F_T = \vec{F} \cdot \vec{T}$$

[See HW 5.1]

The integral of $\vec{F}$ along $\vec{r}(t)$ is

$$\int_C \vec{F} \cdot \vec{T} \, ds = \int \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) \, dt$$

We also obtain a unit vector in the normal direction by rotating $\vec{T}$ 90° clockwise. In terms of the parametrization :

$$\vec{N}(\vec{r}(t)) = \frac{\langle \vec{y}\,'(t), -x'(t) \rangle}{\| \langle \vec{y}\,'(t), -x'(t) \rangle \|} .$$

And the component of $\vec{F}$ in the normal direction is

$$F_N = \vec{F} \cdot \vec{N}$$

We define the "flux of $\vec{F}$ across the curve $\vec{r}(t)$" as the integral of the normal component:

$$\int_C \vec{F} \cdot \vec{N}\, ds = \int \vec{F}(\vec{r}(t)) \cdot \langle y'(t), -x'(t) \rangle\, dt$$

Meaning: How much is $\vec{F}$ pointing perpendicular to (specifically, to the right of) the curve?

To understand the flux form of Green's Theorem, suppose that $\vec{F}$ is the velocity field of a fluid (liquid or gas). Then

$$\iint_D \nabla \cdot \vec{F} \, dA = \int_{\partial D} \vec{F} \cdot \vec{N} \, ds$$

how much does the fluid expand in the region D? = how much fluid flows across the boundary ∂D?

That makes sense!

Examples of Divergence:

- Fluid Dynamics: If $\vec{F}$ is the velocity field of a fluid then

$$\nabla \cdot \vec{F} = \text{infinitesimal amount of expansion / contraction at a point}$$

We often assume that the flow is "incompressible":

$$\nabla \cdot \vec{F} = 0.$$

[This is closely related to "conservation of mass": no fluid is created or destroyed.]

- Gauss' Law for electric & gravitational forces.

  - Let $\rho(x,y,z)$ be a distribution of charge and let $\vec{E} : \mathbb{R}^3 \to \mathbb{R}^3$ be the force exerted on a unit point charge by the charges ρ. Then

$$\nabla \cdot \vec{E} = \rho$$

  - Let $\rho(x,y,z)$ be a distribution of mass and let $\vec{g} : \mathbb{R}^3 \to \mathbb{R}^3$ be the force exerted on a unit point mass by the masses ρ. Then

$$\nabla \cdot \vec{g} = -\rho$$

See HW 5.5 for a 2D example.

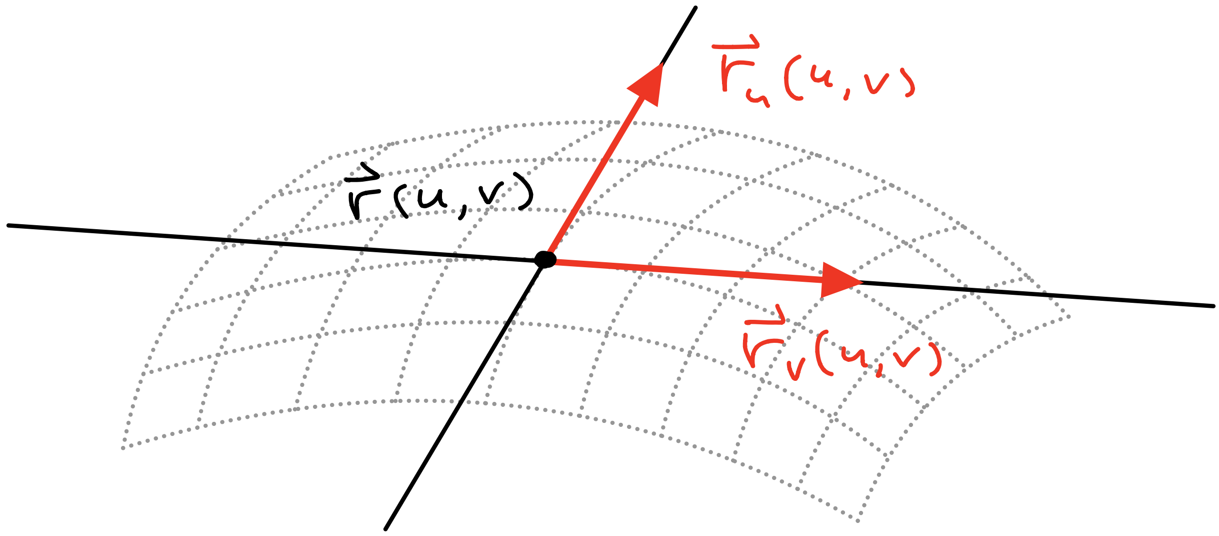
To apply these ideas in 3D we need to discuss "flux across a 2D surface in $\mathbb{R}^3$".

Goal: Integrate a scalar or vector field over a 2D surface in $\mathbb{R}^3$.

How?

We must first <u>parametrize</u> the surface. We can think of a "parametrized surface in $\mathbb{R}^3$" as a function $\vec{r}: \mathbb{R}^2 \to \mathbb{R}^3$:

$$\vec{r}(u,v) = \langle x(u,v),\, y(u,v),\, z(u,v) \rangle.$$



At each point $\vec{r}(u,v)$ we have two basic "velocity vectors"

$$\vec{r}_u = \langle x_u, y_u, z_u \rangle$$
$$\vec{r}_v = \langle x_v, y_v, z_v \rangle$$

The area of a tiny parallelogram near the point $\vec{r}(u,v)$ is the length of a cross product:

$$\underbrace{dS}_{\text{tiny piece of area on the surface}} = \| (\underbrace{\vec{r}_u\, du}) \times (\underbrace{\vec{r}_v\, dv}) \|$$

vectors generating a tiny parallelogram on the surface

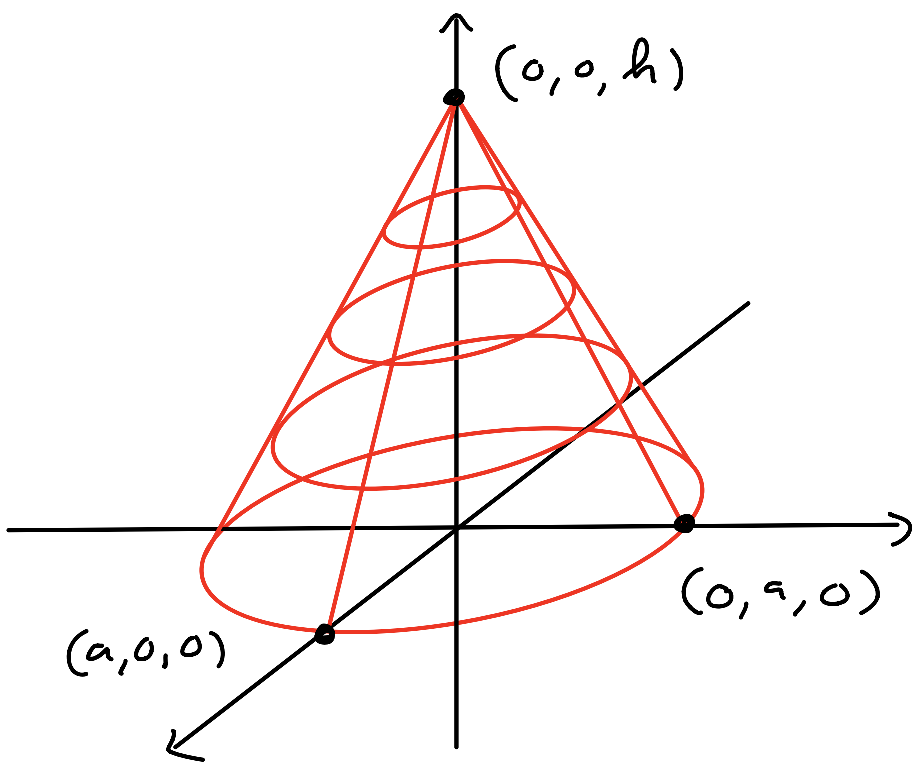
$$= \|\vec{r}_u \times \vec{r}_v\| \, du\, dv$$

This is how we compute the surface area of a parametrized 2D surface in $\mathbb{R}^3$:

$$\text{surface area} = \iint dS$$

$$= \iint \|\vec{r}_u \times \vec{r}_v\| \, du\, dv$$

Example: Surface area of a cone with height h & radius a:



To parametrize the surface it is convenient to use polar coordinates:

$$\vec{r}(r,\theta) = \left\langle r\cos\theta,\ r\sin\theta,\ \frac{h}{a}(a-r) \right\rangle$$

$$\vec{r}_r = \langle \cos\theta,\ \sin\theta,\ -h/a \rangle$$

$$\vec{r}_\theta = \langle -r\sin\theta,\ r\cos\theta,\ 0 \rangle$$

$$\vec{r}_r \times \vec{r}_\theta = \left\langle \frac{h}{a} r\cos\theta,\ \frac{h}{a} r\sin\theta,\ r \right\rangle$$

$$= r \left\langle \frac{h}{a}\cos\theta,\ \frac{h}{a}\sin\theta,\ 1 \right\rangle$$

$$\| \vec{r}_r \times \vec{r}_\theta \| = r\sqrt{\left(\frac{h}{a}\right)^2 + 1}$$

So the surface area is

$$\iint_{\text{cone}} dS = \iint \| \vec{r}_r \times \vec{r}_\theta \| \, dr\, d\theta$$

$$= \iint r\sqrt{\left(\frac{h}{a}\right)^2 + 1}\ dr\, d\theta$$

$$= \sqrt{\left(\frac{h}{a}\right)^2 + 1} \int_0^{2\pi} d\theta \int_0^a r\, dr$$

$$= 2\pi \cdot \frac{1}{2} a^2 \cdot \sqrt{\left(\frac{h}{a}\right)^2 + 1}$$

$$= \pi a \sqrt{h^2 + a^2}$$

[See page 764 of the textbook.]

Finally, let's check that our method gives the correct formula for the surface area of a sphere of radius a.

Let's use spherical coordinates:

$$\vec{r}(\theta,\varphi) = \langle a\cos\theta \sin\varphi, a\sin\theta \sin\varphi, a\cos\varphi \rangle$$

where $0 \le \theta \le 2\pi$,

$0 \le \varphi \le \pi$.

$$\vec{r}_\theta = \langle -a\sin\theta \sin\varphi, a\cos\theta \sin\varphi, 0 \rangle$$

$$\vec{r}_\varphi = \langle a\cos\theta \cos\varphi, a\sin\theta \cos\varphi, -a\sin\varphi \rangle$$

$\vec{r}_\theta \times \vec{r}_\varphi$

$= \langle -a^2\cos\theta \sin^2\varphi,\ a^2 \sin\theta \sin^2\varphi,\ -a^2 \sin\varphi\cos\varphi \rangle$

$\|\vec{r}_\theta \times \vec{r}_\varphi\| =$ computations

$= a^2 \sin\varphi$.

So the surface area is

$$\iint_{\text{sphere}} dS = \iint \|\vec{r}_\theta \times \vec{r}_\varphi\| \, d\theta \, d\varphi$$

$$= \int\int a^2 \sin\varphi \, d\theta \, d\varphi$$

$$= a^2 \int_0^{2\pi} d\theta \int_0^{\pi} \sin\varphi \, d\varphi$$

$$= a^2 \cdot 2\pi \cdot (-(-1) + 1)$$

$$= 4\pi a^2 \quad \checkmark$$