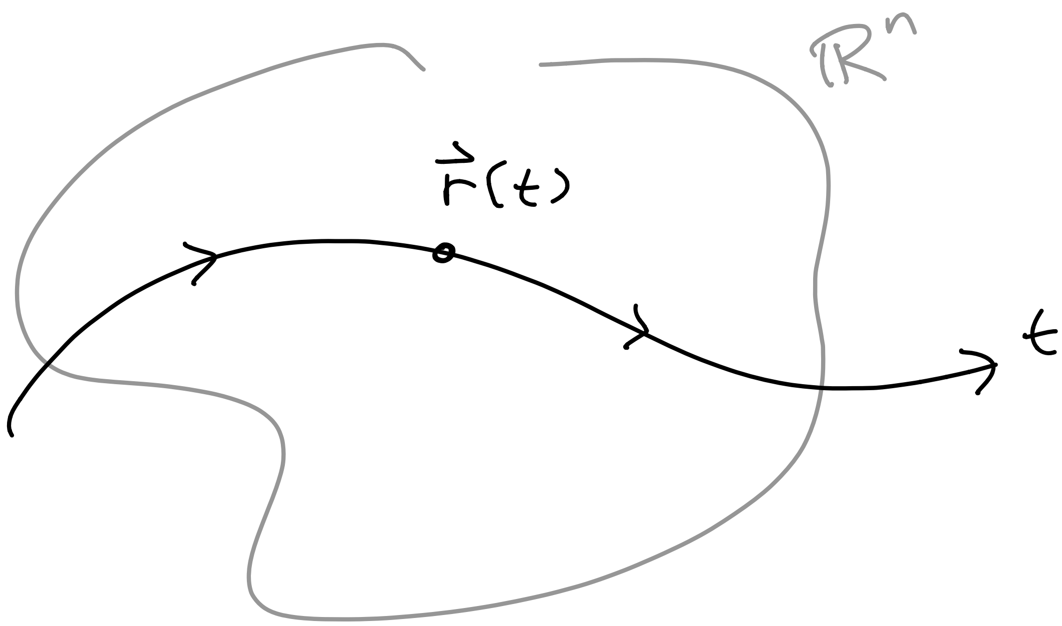
HW3 : TBA

This week : Chapter 4

Chapter 3 was about "vector-valued functions of one variable"

$$\vec{r} : \mathbb{R} \to \mathbb{R}^n$$

We can think of this function as a parametrized curve in n-dim space :



Chapter 4 is about "scalar-valued functions of several variables" :

$$f : \mathbb{R}^n \to \mathbb{R}$$

How should we think of this ?

Generally, we will think of this as a "scalar field", i.e., at each point in n-dimensional space

$$(x_1, x_2, \dots, x_n) \in \mathbb{R}^n$$

we have a single number (a "scalar"):

$$f(x_1, x_2, \dots, x_n) \in \mathbb{R}$$

This number could represent

- temperature
- pressure
- density
- chemical concentration
- etc.

How can we visualize a scalar field ?

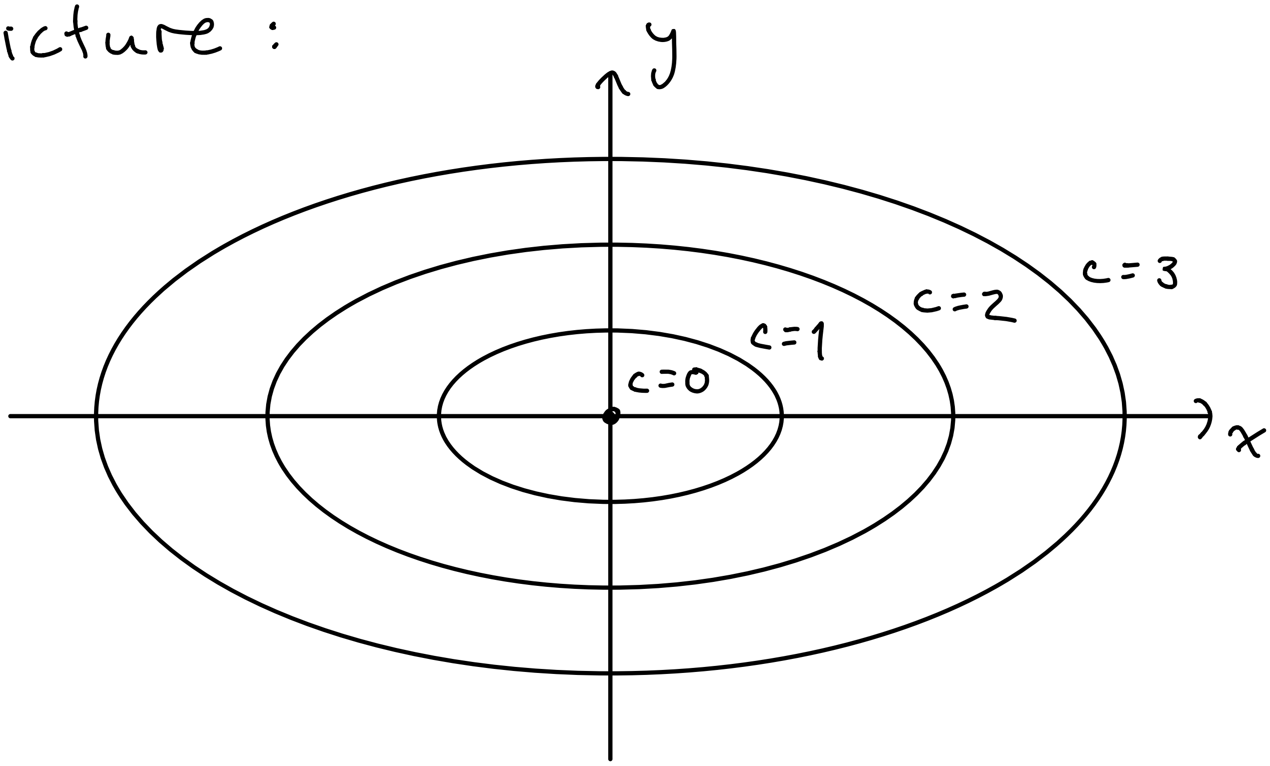
Example: Suppose the temperature at a point $(x,y) \in \mathbb{R}^2$ satisfies

$$f(x,y) = \left(\frac{x}{2}\right)^2 + y^2 .$$

For a fixed temperature c, the set of points with this temperature forms an ellipse with equation

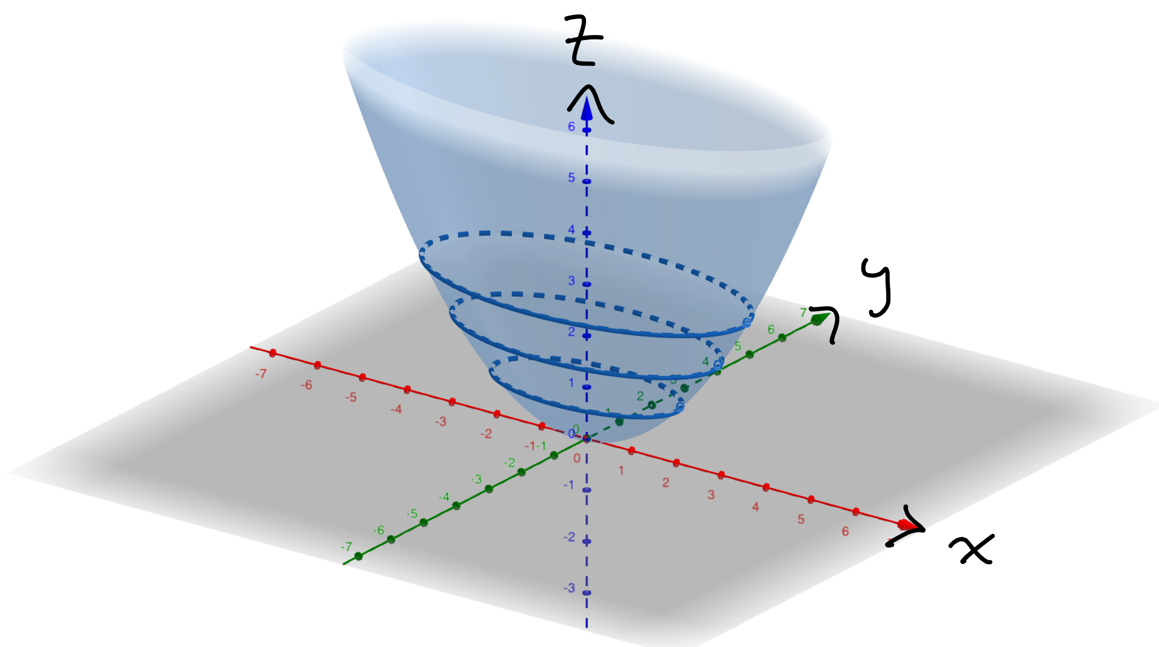
$$f(x,y) = c$$
$$\left(\frac{x}{2}\right)^2 + y^2 = c$$

Picture:



These ellipses are called "isotherms", or "curves of constant temperature". Every point has some temperature c, so these curves cover the whole plane.

If we think of temperature as a "third variable", say z, then we can view this "distribution of temperature in $\mathbb{R}^2$" as a 2D surface living in $\mathbb{R}^3$:



Here we let "height above xy-plane" represent "temperature".

Jargon: This 2D surface in 3D is "the graph" of the function $f(x,y)$.

[Warning: The "graph" of a temperature distribution $f(x,y,z)$ in $\mathbb{R}^3$ would be a "3D surface living in 4D space", which isn't very helpful, so we won't consider graphs of $f(x_1,\dots,x_n)$ when $n \geq 3$.]

Big Question in Chapter 4:

How should we define "the derivative" of a "scalar field", i.e., a function of several variables $f: \mathbb{R} \to \mathbb{R}^n$?

Ideas :

- The rate of change of (e.g.) temperature.

The problem with this idea is that the rate of change of temperature depends on your direction & speed !

Let's Fix the idea :

- The rate of change of temperature in the direction of a specific vector (i.e., your velocity vector).

But we also think of "derivative" as "slope of the tangent" :

- "The derivative" of a function $f(x,y)$ should tell us about the <u>tangent planes</u> to the surface

$$z = f(x,y).$$

Amazingly, it turns out that all of these ideas are encoded in the "gradient vector" of the scalar field.

Definition: Given a function f in n variables

$$f(x_1, x_2, \dots, x_n),$$

we define the <u>gradient vector</u>

$$\nabla f = \left\langle \frac{df}{dx_1}, \frac{df}{dx_2}, \dots, \frac{df}{dx_n} \right\rangle$$

Example: Let

$$f(x,y) = \left(\frac{x}{2}\right)^2 + y^2 = \frac{1}{4}x^2 + y^2,$$

so that $\frac{df}{dx} = \frac{1}{4} \cdot 2x + 0 = \frac{1}{2}x$

& $\frac{df}{dy} = 0 + 2y = 2y$

Then the gradient vector at the point (x,y) is

$$\nabla f(x,y) = \left\langle \frac{1}{2}x\,,\, 2y \right\rangle$$

Remarks :

- ∇ is called "nabla".
- The vector $\nabla f(x,y)$ changes from point to point. Later we will say that ∇f is a "vector field"

$$\nabla f : \mathbb{R}^2 \to \mathbb{R}^2$$

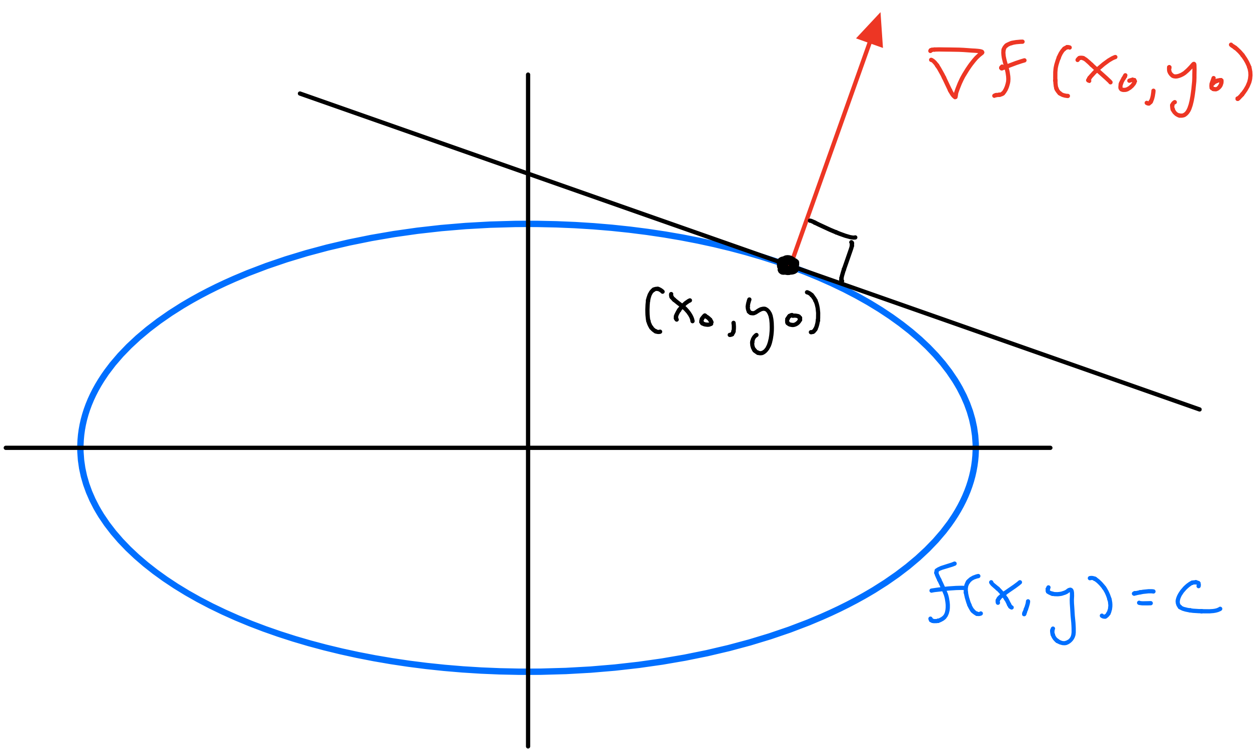
[Examples of vector fields : force field, velocity field, electric field, ...]

It turns out that gradient vectors are perpendicular to "level curves".

Example: The gradient vector of $f(x,y) = \frac{1}{4}x^2 + y^2$ at the point (x_0, y_0) is

$$\nabla f(x_0, y_0) = \langle \tfrac{1}{2}x_0, 2y_0 \rangle$$

If $f(x_0,y_0) = c$ [e.g. the temp. at point $(x_0,y_0) = c$] then we obtain the following picture:



The gradient vector at (x_0,y_0) is $\perp$ to the "isotherm" at (x_0,y_0).

[Idea for later: the gradient vector is the "direction of largest increase of temperature".]

Thus we can use the gradient vector to find the equation of the tangent line at (x_0, y_0):

Let $\langle a, b \rangle = \nabla f(x_0, y_0) = \langle \frac{1}{2}x_0, 2y_0 \rangle$ so the equation of the tangent line is

$$a(x - x_0) + b(y - y_0) = 0$$

$$\frac{1}{2}x_0(x - x_0) + y_0(y - y_0) = 0$$

On HW 1.3 we took the point

$$(x_0, y_0) = (\sqrt{2}, \sqrt{2}/2)$$

$$\nabla f(x_0, y_0) = \langle \tfrac{1}{2}x_0, 2y_0 \rangle$$

$$= \langle \sqrt{2}/2, \sqrt{2} \rangle$$

So the tangent line at $(\sqrt{2}, \sqrt{2}/2)$ is

$$\frac{\sqrt{2}}{2}(x-\sqrt{2}) + \sqrt{2}\left(y - \sqrt{2}/2\right) = 0$$

$$\frac{1}{2}x - \frac{\sqrt{2}}{2} + y - \frac{\sqrt{2}}{2} = 0$$

$$y = -\frac{1}{2}x + \sqrt{2},$$

just as in HW 1 ✓