HW5 is up; due Tues.

Review:

- A vector field $\vec{F}$ is called "conservative" if it is a gradient,

$$\vec{F} = \nabla f.$$

That is, if $\vec{F}$ has an "antiderivative".

[Example: If $\vec{F} = -\nabla f$ is a conservative force field then the scalar field f is called "potential energy".]

- Given vector field $\vec{F} = \langle P, Q, R \rangle$ in 3D space we define the "curl"

$$\nabla \times \vec{F} = \text{“} \langle \partial_x, \partial_y, \partial_z \rangle \times \langle P, Q, R \rangle \text{”}$$

$$= \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle,$$

which is also a vector field in 3D.

Given any 2D vector field $\vec{F} = \langle P, Q \rangle$ we can also define the "curl" of $\vec{F}$,

$$\text{curl}(\vec{F}) = Q_x - P_y,$$

which is a <u>scalar</u> field!

Remarks:

- We can think of $\vec{F} = \langle P, Q \rangle$ as the shadow of $\vec{F} = \langle P, Q, 0 \rangle$, so

$$\nabla \times \vec{F} = \langle 0, 0, Q_x - P_y \rangle.$$

Idea: The curl of a 2D vector field $\langle P, Q \rangle$ points "up" into the z-direction.

- There is no concept of "curl" outside of 2D & 3D space. ///

- "Cross-Partial" Property of Conservative Vector Fields.

In $\mathbb{R}^3$:

vector

$\vec{F}$ is cons. $\iff$ $\nabla \times \vec{F} = \langle 0, 0, 0 \rangle$

$$\iff \begin{cases} R_y = Q_z \\ P_z = R_x \\ Q_x = P_y \end{cases}$$

In $\mathbb{R}^2$:

scalar

$\vec{F}$ is cons. $\iff$ $\text{curl}(\vec{F}) = 0$

$\iff Q_x = P_y$

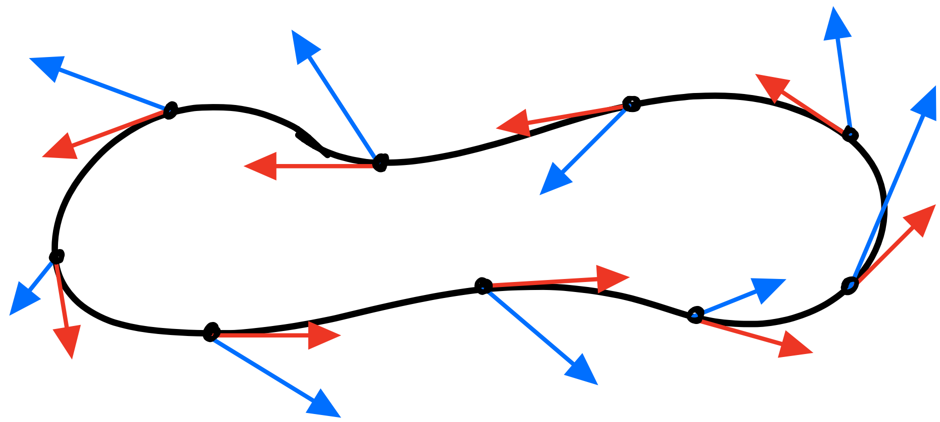
Intuition: If $\vec{F} = \nabla f$ then

$$\oint_{\text{loop}} \vec{F} \cdot \vec{T}\, ds = 0 \quad \text{for } \underline{\text{any}} \text{ loop.}$$

If $\text{curl}(\vec{F}) \neq 0$ then we can find some loop C such that

$$\oint_C \vec{F} \cdot \vec{T}\, ds > 0$$

Picture :



This vector field "curls" with the loop.

Example : $\vec{F}(x,y) = \langle -y, x \rangle$.

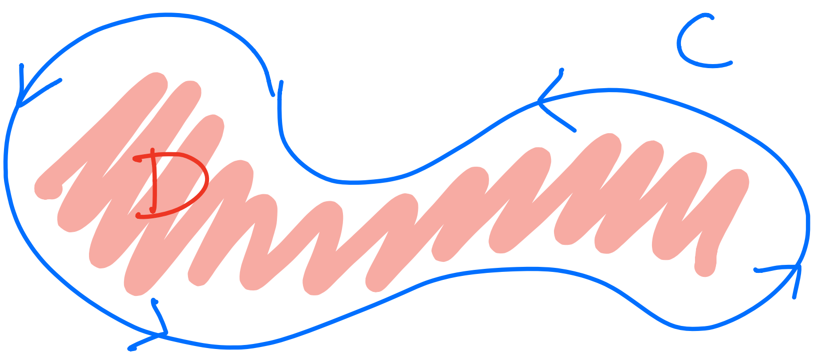
We saw that this vector field "curls" counterclockwise. And, indeed, we have

$$\begin{aligned} \operatorname{curl}(\vec{F}) &= Q_x - P_y \\ &= \partial_x(x) - \partial_y(-y) \\ &= 1 + 1 = 2 > 0 \end{aligned}$$

The next theorem will make this intuition precise.

Green's Theorem:

Let C be a "simple closed curve" (no self-intersections), oriented counterclockwise. Let D be the interior 2D region:



[Mnemonic: Region D is always "to the left" of the curve C.]

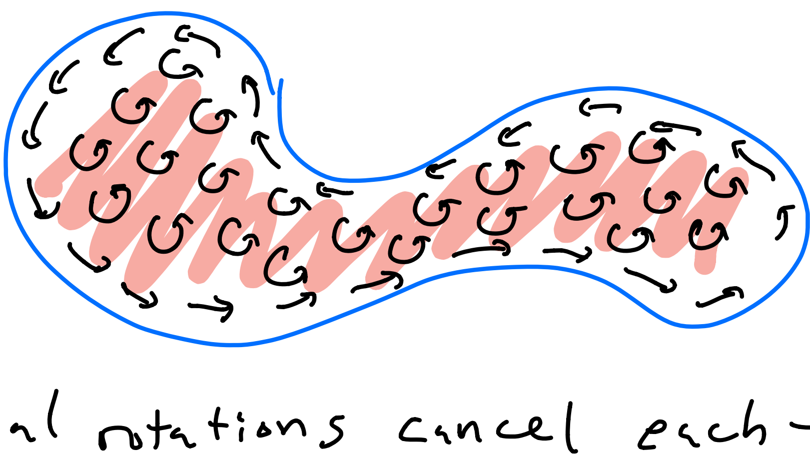
If $\vec{F} = \langle P, Q \rangle$ is a 2D field defined at every point of D, then

$$\underbrace{\iint_D \operatorname{curl}(\vec{F})\,dA}_{\text{amount that } \vec{F} \text{ is ``curling'' inside the loop}} = \underbrace{\oint_C \vec{F}\cdot\vec{T}\,ds}_{\text{amount that } \vec{F} \text{ points ``along'' the loop}}$$

This is the precise meaning of curl:

"$\operatorname{curl}(\vec{F})\,dA$" = how much is $\vec{F}$ pointing along a tiny c.c.w. loop?

The proof is hard, but here is the idea:



All the internal rotations cancel each-other and all that's left is the circulation along the boundary. ///

Some other notations:

$$\iint \operatorname{curl}(\vec{F})\,dA = \oint \vec{F}\cdot d\vec{r}$$

$$\iint (Q_x - P_y)\,dx\,dy = \oint \langle P, Q\rangle \cdot \left\langle \frac{dx}{\cancel{dt}}, \frac{dy}{\cancel{dt}} \right\rangle \cancel{dt}$$

$$\boxed{\iint (Q_x - P_y)\,dx\,dy = \oint P\,dx + Q\,dy}$$

This is just a notation. It means exactly the same thing.

One more observation:

If $\operatorname{curl}(\vec{F}) = 0$ everywhere, then for any loop C we have

$$\oint \vec{F}\cdot\vec{T}\,ds = \iint 0\,dA = 0,$$

so that $\vec{F}$ is conservative.

Thus the "cross-partial" property is proved ✓

As with any "Fundamental Theorem

of Calculus", Green's Theorem can be used to simplify computations.

Example: Let C be the c.c.w. oriented perimeter of the rectangle

$$0 \leq x \leq 2,$$
$$0 \leq y \leq 3.$$

Compute the circulation of the vector field $\vec{F} = \langle P, Q \rangle = \langle x^2 y, y-3 \rangle$ around C.

To do this directly you need to parametrize all four sides of the rectangle and compute four line integrals. Instead, we will use Green's Theorem and integrate the curl $Q_x - P_y = 0 - x^2$ not conservative over the interior of the rectangle:

$$\oint_C \vec{F} \cdot \vec{T}\, ds = \iint_D \operatorname{curl}(\vec{F})\, dA$$

$$= \iint_D -x^2\, dx\, dy$$

$$= \int_0^3 \int_0^2 -x^2\, dx\, dy$$

$$= \int_0^3 -\frac{8}{3}\, dy$$

$$= 3\left(-\frac{8}{3}\right) = -8.$$

The force drained 8 units of KE while you traveled ccw around the perimeter. Or, your airplane needed 8 gallons of fuel to oppose the wind $f(x,y) = \langle x^2 y, y-3 \rangle$.

But be careful!

WARNING: Consider the field

$$\vec{F}(x,y) = \frac{1}{x^2+y^2}\langle -y, x\rangle$$

$$= \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$$

We have

$$Q_x = \partial_x \left(\frac{x}{x^2+y^2} \right)$$

$$= \frac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2}$$

$$= (y^2-x^2) \big/ (x^2+y^2)^2$$

and

$$P_y = \partial_y \left(\frac{-y}{x^2+y^2} \right)$$

$$= \frac{(x^2+y^2)(-1)-(-y)(2y)}{(x^2+y^2)^2}$$

$$= (y^2-x^2)\big/(x^2+y^2)^2$$

$$= Q_x \quad (\text{surprise!})$$

So that $\text{curl}(\vec{F}) = Q_x - P_y = 0$.

However, I claim that

$$\oint_{\substack{\text{unit}\\ \text{circle}}} \vec{F}\cdot\vec{T}\,ds \neq 0.$$

To ~~see~~ this, we can use the standard parametrization:

$$\vec{r}(t) = \langle \cos t, \sin t\rangle, \quad t = 0 \text{ to } 2\pi.$$

$$\vec{F}(\vec{r}(t)) = \frac{1}{\cos^2 t + \sin^2 t}\langle -\sin t, \cos t\rangle$$

$$= \langle -\sin t, \cos t\rangle$$

$$\vec{r}\,'(t) = \langle -\sin t, \cos t \rangle$$

and hence

$$\oint_{\vec{r}} \vec{F} \cdot d\vec{r} = \int \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

$$= \int_0^{2\pi} (\sin^2 t + \cos^2 t)\, dt$$

$$= \int_0^{2\pi} 1\, dt$$

$$= 2\pi \neq 0.$$

What went wrong?

Green's Theorem assumes that curl($\vec{F}$) exists at every point inside the loop. But this field $\vec{F}(x,y) = \langle -y, x \rangle / (x^2+y^2)$, and hence its curl, is not defined at the origin. So Green's Theorem doesn't apply to this calculation.

Nevertheless, Green's Theorem does give us a simplification. For any (simple, closed, ccw) loop C I claim that

$$\oint_C \vec{F} \cdot \vec{T}\, ds = \begin{cases} 0 & \text{if } C \text{ does not contain } (0,0) \\ 2\pi & \text{if } C \text{ contains } (0,0) \end{cases}$$

So this integral doesn't depend on the <u>shape</u> of the loop, but it <u>does</u> detect whether the loop goes around the origin. Strange!

I'll explain this next time ...