Schedule :

HW5 due Tues, June 22

Quiz 5 on Wed, June 23

Thurs, June 24, no class

Final Project due Fri, June 25

[Write a point form summary of what you learned in this class : important formulas, definitions, etc.]

Chapter 6 : Vector Calculus.

This chapter is more sophisted and sadly we don't have much time left.

Topics :

(1) Differentiation of vector fields in $\mathbb{R}^2$ and $\mathbb{R}^3$.

(2) Integration of scalar and vector

fields along 1D curves in $\mathbb{R}^2$ and $\mathbb{R}^3$ & along 2D surfaces in $\mathbb{R}^3$.

(3) The "fundamental theorems of vector calculus" connecting (1) & (2).

Recall from Calc I that

$$\int_a^b f'(t)\,dt = f(b) - f(a)$$

Indeed, this is how we <u>compute</u> integrals. Before the discovery of this theorem integrals were extremely difficult. (Now they are somewhat less difficult...)

The general idea is that "integration reverses differentiation".

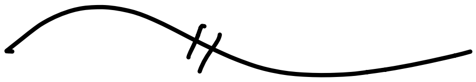
This idea still holds in $\mathbb{R}^2$ & $\mathbb{R}^3$:

- $\displaystyle\int_{\text{curve } \vec{r}} \nabla f \cdot d\vec{r} = f(\text{endpoint of } \vec{r}) - f(\text{starting point of } \vec{r})$

- $\displaystyle\iint_{\text{surface } S} \nabla \times \vec{F} \cdot d\vec{S} = \int_{\substack{\text{boundary curve} \\ \vec{r} \text{ of } S}} \vec{F} \cdot d\vec{r}$

- $\displaystyle\iiint_{\substack{\text{solid region} \\ E}} \nabla \cdot \vec{F}\, dV = \iint_{\substack{\text{boundary surface} \\ S \text{ of region } E}} \vec{F} \cdot d\vec{S}$

My goal is to state the definitions, explain the meanings, and compute <u>some</u> (but sadly not enough) examples of these "fundamental theorems".



Integration along a curve $\vec{r}(t)$:

Recall that

$$\text{arc length} = \int \text{speed}\, dt$$
$$= \int \|\vec{r}\,'(t)\|\, dt$$

We can also write this as

$$\text{arc length} = \int ds$$
$$= \int \text{length of a tiny piece of the curve},$$

where

$$\underbrace{ds}_{\text{tiny distance}} = \underbrace{\|\vec{r}\,'(t)\|}_{\text{speed}}\, \underbrace{dt}_{\text{tiny time}}$$

The "same definition" works for any scalar field f. To "integrate f along the curve $\vec{r}(t)$":

$$\boxed{\int f\,ds = \int f(\vec{r}(t))\,\|\vec{r}\,'(t)\|\,dt}$$

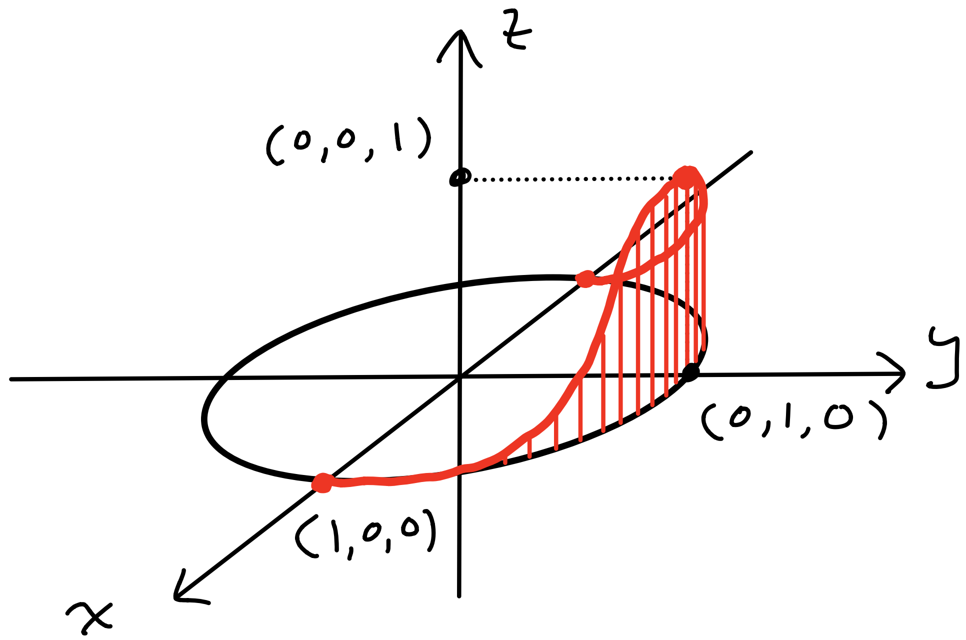
"line integral of a scalar field"

What does it mean?

If we think of $f(x,y)$ as "height above the xy-plane" then the integral is the area of the "vertical wall" above the curve $\vec{r}(t) = \langle x(t), y(t) \rangle$ and below the surface $z = f(x,y)$.

Example: Find the area of the vertical wall above the circle $x^2+y^2=1$ and below the parabolic surface $z = x^2$, where $x \geq 0$.

Picture:



Idea : The total area is

$$\text{area} = \int \text{area of skinny rectangle}$$

$$= \int \underset{\text{height}}{x^2} \, \underset{\text{length of base}}{ds}$$

To compute it we need to parametrize the base curve $x^2+y^2=1$ where $x \geq 0$. Any parametrization will do. Let's take

$$\vec{r}(t) = \langle \cos t, \sin t \rangle$$

where t goes from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$.

According to our definition:

$$\text{area of the wall} = \int x^2 \, ds$$

$$= \int_{-\pi/2}^{\pi/2} x(t)^2 \, \|\vec{r}\,'(t)\| \, dt$$

$$= \int_{-\pi/2}^{\pi/2} \cos^2 t \sqrt{\sin^2 t + \cos^2 t} \; dt$$

$$= \int_{-\pi/2}^{\pi/2} \cos^2 t \, dt$$

$$= \left(\frac{t}{2} + \frac{1}{4} \sin(2t) \right)_{-\pi/2}^{\pi/2}$$

$$= \frac{\pi}{2} + \frac{1}{4} \left(\overset{0}{\cancel{\sin(\pi)}} - \overset{0}{\cancel{\sin(-\pi)}} \right)$$

$= \pi/2$.

But a scalar field need not represent "height". If $f(x,y)$ is "muddiness" of a field at point (x,y) the

$$\int f(\vec{r}(t))\,\|\vec{r}\,'(t)\|\,dt$$

is the total amount of mud that you pick up on your journey.

More interesting: We can integrate a vector field $\vec{F} : \mathbb{R}^n \to \mathbb{R}^n$ over a path $\vec{r} : \mathbb{R} \to \mathbb{R}^n$.

Then instead of

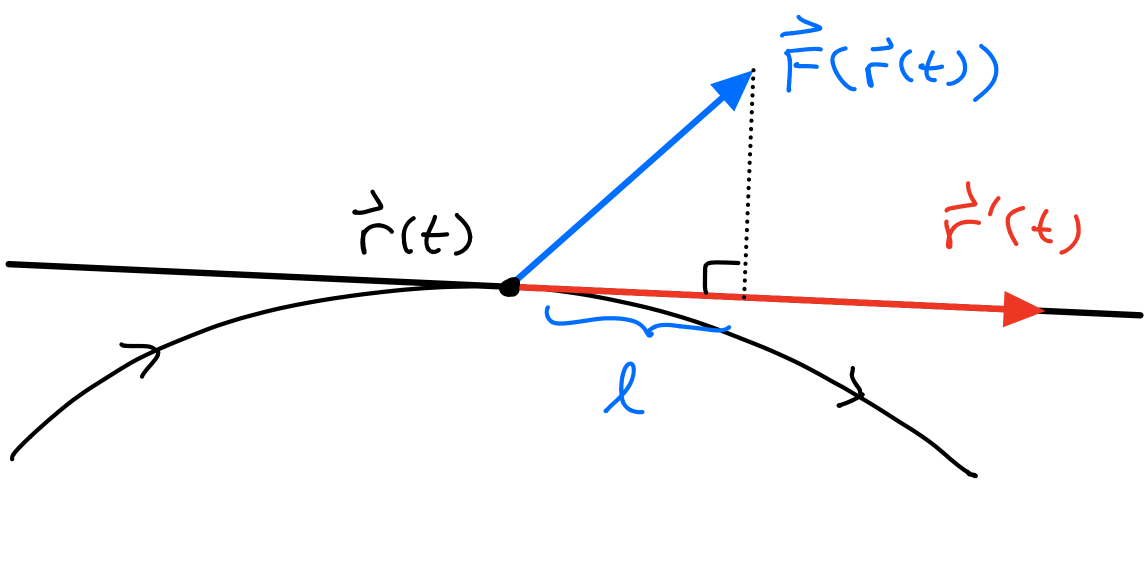
$$\int f\,ds = \int f(\vec{r}(t))\,\|\vec{r}\,'(t)\|\,dt$$

we will define

$$\boxed{\int \vec{F}\cdot d\vec{r} = \int \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt}$$

What does it mean ?

Idea : How much of the vector field is pointing in the same direction as your velocity ?



Let ℓ denote the component of of the vector $\vec{F}(\vec{r}(t))$ pointing in the direction of the velocity.

We allow ℓ to be negative!

[Sometimes the "force" $\vec{F}$ opposes the motion of the curve.]

Geometric Theorem (see HW 5):

$$\ell = \vec{F}(\vec{r}(t)) \cdot \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}$$

This is a unit (length 1) vector in the direction of the tangent.

The textbook calls it

$$\vec{T}(\vec{p}) = \vec{T}(\vec{r}(t)) = \frac{r'(t)}{\|\vec{r}\,'(t)\|}$$

Note that $\vec{T}(\vec{p})$ only depends on the point $\vec{p}$ and the shape of the curve C. It does not depend on the choice of parametrization $\vec{r}(t)$. For this reason, the line integral is often written as

$$\int_C \vec{F} \cdot \vec{T} \, ds$$

However, when we compute the integral then we must choose a parametrization $\vec{r}(t)$ for C. Then we obtain

$$\vec{T}(\vec{r}(t)) = \vec{r}\,'(t) / \|\vec{r}\,'(t)\|,$$

$$ds = \|\vec{r}\,'(t)\| \, dt,$$

and hence

$$\int_C \vec{F} \cdot \vec{T}\, ds$$

$$= \int_{\vec{r}} \vec{F}(\vec{r}(t)) \cdot \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|} \|\vec{r}'(t)\|\, dt$$

$$= \int_{\vec{r}} \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\, dt$$

If that isn't complicated enough, our book also uses the notation

$$d\vec{r} = \vec{r}'(t)\, dt,$$

so the integral of the vector field $\vec{F}$ along the parametrized curve $\vec{r}$ can also be written as

$$\int_{\vec{r}} \vec{F} \cdot d\vec{r}$$

I apologize that the notation is very complicated. It is most important to understand the meaning of the definition:

$\int \vec{F} \cdot d\vec{r} =$ "how much is the vector field $\vec{F}$ pointing in the same direction as the curve $\vec{r}$?"

But why do we care?

Answer: ENERGY!

I'll explain this next time.