HW 4 due Mon 11:40am

Quiz 4 on Mon at 11:40am.

Today: "Finish" Chapter 5.

A double integral can represent a 3D volume "below a surface":

$$\text{vol} = \iint \underbrace{f(x,y)}_{\text{height}} \underbrace{dx\,dy}_{\text{area of base}}$$

But it can also represent the mass or the area of a 2D region:

$$\text{mass} = \iint \underbrace{\rho(x,y)\,dx\,dy}_{\text{mass of a little rectangle}}$$

For area: If D is a 2D region with uniform density ρ (units of mass per unit of area) then

$$\text{mass}(D) = \rho \ \text{area}(D)$$

$$\iint \underbrace{\rho \, dx\,dy}_{\text{mass of a little rectangle}} = \rho \iint \underbrace{1 \, dx\,dy}_{\text{area of a little rectangle}}$$

[Remark: "Σ" and "$\int$" are "S".

"Σ" = discrete sum

"$\int$" = continuous sum (integral).]

A triple integral can represent the "hypervolume of a 4D region" but we won't use it for this. Instead we will use triple integrals to represent 3D mass, 3D volume, etc. [We can integrate any scalar field over a 3D region.]

Examples :

$$\iiint dx\,dy\,dz = \text{volume}$$

$$\iiint \rho(x,y,z)\,dx\,dy\,dz = \text{mass}$$

Example of Volume : Compute the volume of a rectangular box

$$a_1 \leq x \leq a_2$$
$$b_1 \leq y \leq b_2$$
$$c_1 \leq z \leq c_2$$

$$\text{volume} = \iiint 1\,dx\,dy\,dz$$

$$= \iint \left(\int_{a_1}^{a_2} 1\,dx \right) dy\,dz$$

$$= \int \left(\int_{b_1}^{b_2} (a_2 - a_1)\,dy \right) dz$$

$$= \int_{c_1}^{c_2} (a_2 - a_1)(b_2 - b_1)\,dz$$

$$= (a_2 - a_1)(b_2 - b_1)(c_2 - c_1) \quad \checkmark$$

Harder : Compute the mass of the box

$$0 \le x \le 1$$
$$0 \le y \le 2$$
$$0 \le z \le 3$$

if the density distribution is

$$\rho(x,y,z) = x + z .$$

$$\text{mass} = \iiint \rho \, dx\,dy\,dz$$

$$= \iint \left(\int_0^1 (x+z)\,dx \right) dy\,dz$$

$$= \iint \left(\frac{1}{2}x^2 + zx \right)\Big|_{x=0}^{x=1} dy\,dz$$

$$= \int \left(\int_0^2 \left(\frac{1}{2} + z \right) dy \right) dz$$

$$= \int \left(\frac{1}{2} y + z y \right)_{y=0}^{y=2} dz$$

$$= \int_0^3 (1 + 2z) \, dz$$

$$= \left(z + z^2 \right)_{z=0}^{z=3}$$

$$= 3 + 9 = 12 \text{ units of mass.}$$

Remark: The order of integration doesn't matter, so there are 6 different ways to compute this!

Center of Mass?

Let E be a solid 3D region with density $\rho(x, y, z)$.

The center of mass has coordinates

$$(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{M_{yz}}{m}, \frac{M_{xz}}{m}, \frac{M_{xy}}{m} \right)$$

where

$$m = \iiint \rho \, dV = \text{mass}$$

$$\left.\begin{aligned} M_{yz} &= \iiint x \rho \, dV \\ M_{xz} &= \iiint y \rho \, dV \\ M_{xy} &= \iiint z \rho \, dV \end{aligned}\right\} \text{“moments about the coordinate planes”}$$

In our example:

$$m = \iiint (x+z) \, dx \, dy \, dz = 12$$

$$M_{yz} = \iiint x(x+z) \, dx \, dy \, dz$$

$$= \iint \left(\int_0^1 (x^2 + xz) \, dx \right) dy \, dz$$

$$= \iint \left(\frac{1}{3}x^3 + \frac{1}{2}x^2 z \right)_0^1 \, dy \, dz$$

$$= \int \left(\int_0^2 \left(\frac{1}{3} + \frac{1}{2} z \right) dy \right) dz$$

$$= \int \left(\frac{1}{3}y + \frac{1}{2} zy \right)_0^2 \, dz$$

$$= \int_0^3 \left(\frac{2}{3} + z \right) dz$$

$$= \left(\frac{2}{3} z + \frac{1}{2} z^2 \right)_0^3$$

$$= 2 + \frac{9}{2} = 13/2$$

units ? who cares

Similar computations give

$M_{xz} = 12$ & $M_{xy} = 45/2$.

So the center of mass of the box is

$$(\bar{x}, \bar{y}, \bar{z}) = \left(M_{yz}/m, \; M_{xz}/m, \; M_{xy}/m \right)$$

$$= \left(\frac{13/2}{12}, \frac{12}{12}, \frac{45/2}{12} \right)$$

$$\approx (0.54, 1.00, 1.86)$$

For 2D integrals we have Cartesian & polar coordinates:

$$dA = dx\,dy = r\,dr\,d\theta.$$

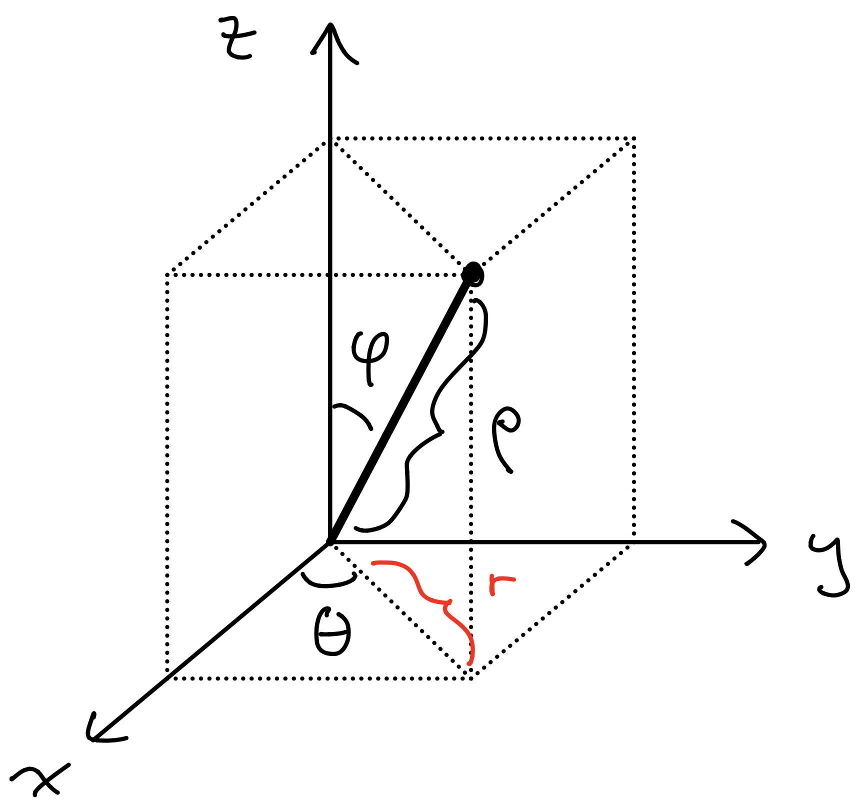
For 3D integrals we have Cartesian, cylindrical & spherical coordinates. We already saw cylindrical coordinates (r, θ, z):

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

The "volume element" is

$$dV = \underbrace{r\,dr\,d\theta}_{\text{area of the base}}\ \underset{\text{height}}{dz}$$

Next we discuss spherical coordinates (ρ, θ, φ):



Conversion to Cylindrical :

$$\begin{cases} r = \rho \sin\varphi \\ \theta = \theta \\ z = \rho\cos\varphi \end{cases}$$

Conversion to Cartesian :

$$\begin{cases} x = r\cos\theta = \rho\sin\varphi\cos\theta \\ y = r\sin\theta = \rho\sin\varphi\sin\theta \\ z = \rho\cos\varphi \end{cases}$$

What is the "volume element" in spherical coordinates ?

$$\begin{aligned} dV &= dx\,dy\,dz \\ &= r\,dr\,d\theta\,dz \\ &= \boxed{?}\,d\rho\,d\theta\,d\varphi \end{aligned}$$

This time it's difficult to give

a convincing heuristic derivation.

[Our textbook gives an <u>unconvincing</u> derivation!]

The correct formula is given by the "Jacobian determinant"

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)} = \det\begin{pmatrix} dx/d\rho & dy/d\rho & dz/d\rho \\ dx/d\theta & dy/d\theta & dz/d\theta \\ dx/d\varphi & dy/d\varphi & dz/d\varphi \end{pmatrix}$$

$$= \det\begin{pmatrix} \sin\varphi\cos\theta & \sin\varphi\sin\theta & \cos\varphi \\ -\rho\sin\varphi\sin\theta & \rho\sin\varphi\cos\theta & 0 \\ \rho\cos\varphi\cos\theta & \rho\cos\varphi\sin\theta & -\rho\sin\varphi \end{pmatrix}$$

$= -\rho^2 \sin\varphi$ (computer)

Conclusion :

$$\boxed{dV = \rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi}$$

take $0 \le \varphi \le \pi$ so volume is ≥ 0

Remark: The same formula gives the "stretch factor" between any two coordinate systems.

For Cartesian to Polar we get

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \det\begin{pmatrix} dx/dr & dy/dr \\ dx/d\theta & dy/d\theta \end{pmatrix}$$

$$= \det\begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix}$$

$$= r\cos^2\theta - (-r\sin^2\theta)$$

$$= r(\cos^2\theta + \sin^2\theta)$$

$$= r$$

This is the true meaning behind our previous notation:

$$\partial(x,y)/\partial(r,\theta) = r$$

$$\text{“} \partial(x,y) = r\,\partial(r,\theta) \text{”}$$

$$\text{“} dx\,dy = r\,dr\,d\theta \text{”}$$

Maybe that wasn't convincing. Let's test it.

Example: Compute the volume of a sphere of radius a, using spherical coordinates. It's easy to parametrize the sphere:

$$0 \leq \rho \leq a$$

$$0 \leq \theta \leq 2\pi$$

$$0 \leq \varphi \leq \pi$$

angle from the vertical is at most $180°$

Then the volume is

$$\text{vol} = \iiint 1\, dV$$

volume of a tiny pice

$$= \iiint \rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi$$

$$= \iint \left(\int_0^a \rho^2 \sin\varphi \, d\rho \right) d\theta \, d\varphi$$

$$= \iint \left(\frac{1}{3}\rho^3 \sin\varphi \right)_{\rho=0}^{\rho=a} d\theta \, d\varphi$$

$$= \int \left(\int_0^{2\pi} \frac{1}{3} a^3 \sin\varphi \, d\theta \right) d\varphi$$

$$= \int_0^{\pi} 2\pi \cdot \frac{1}{3} a^3 \sin\varphi \, d\varphi$$

$$= \frac{2}{3}\pi a^3 \left(-\cos\varphi \right)_{\varphi=0}^{\varphi=\pi}$$

$$= \frac{2}{3}\pi a^3 \left(\underbrace{-\cos(\pi) - (-\cos(0))}_{2} \right)$$

$$= \frac{4}{3}\pi a^3 ,$$

and that is the correct answer.
So I guess the formula

$$dV = \rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi$$

is correct ....

There is an example of cylindrical coordinates on

HW 4, Problem 4.

I will add an example of spherical coordinates in Problem 5.