HW 4 is posted; due Mon.

Quiz 4 is on Tues (note change).

Chapter 5: Multiple Integration.

We have used double integrals to compute volumes. We can think of a function $f(x,y)$ as the height of the surface $z = f(x,y)$ above the point (x,y). Then the double integral is a volume:

$$\text{volume} = \iint \underbrace{f(x,y)}_{\text{height}} \, \underbrace{dx\,dy}_{\text{area of the base}}$$

However, $f(x,y)$ doesn't have to represent height. It could also represent temperature, mass density, energy, or any scalar quantity.

For example, let $\rho(x,y)$ be the "local mass density" in a 2D region D at the point (x,y). Then the double integral is the total mass of the region:

$$\text{mass} = \iint \text{density}\, dA$$

$$\text{mass of } D = \iint_D \rho(x,y)\, dx\, dy$$

When the density is constant, $\rho(x,y) = 1$ unit of mass per unit of area, then the mass is just the same as the <u>area</u>:

$$\begin{aligned} \text{area of } D &= \text{mass of } D \\ &= \iint_D 1\, dx\, dy \end{aligned}$$

Example: Let's derive the formula for the area of a circle with radius a, in two ways.

- Cartesian Coordinates.

We can parametrize the circle by

$$-a \leq x \leq +a$$

$$-\sqrt{a^2-x^2} \leq y \leq +\sqrt{a^2-x^2}$$

Then we have

this is the area of a tiny rectangle

$$\text{area of the circle} = \iint_{\text{circle}} 1 \, dx \, dy$$

$$= \int_{x=-a}^{x=+a} \left(\int_{y=-\sqrt{a^2-x^2}}^{y=+\sqrt{a^2-x^2}} 1 \, dy \right) dx$$

$$= \int_{x=-a}^{x=+a} 2\sqrt{a^2 - x^2}\, dx$$

Trig sub, I guess?

$$x = a \sin\theta$$

$$a^2 - x^2 = a^2(1 - \sin^2\theta)$$

$$= a^2 \cos^2\theta$$

$$+\sqrt{a^2 - x^2} = a\cos\theta \qquad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$

Since

$$x = -a \longrightarrow +a$$

$$\sin\theta = -1 \longrightarrow +1$$

$$\theta = -\frac{\pi}{2} \longrightarrow +\frac{\pi}{2}$$

and $dx = a\cos\theta\, d\theta$, we obtain

$$\text{area} = \int_{-\pi/2}^{\pi/2} 2\, a\cos\theta\, a\cos\theta\, d\theta$$

$$= 2a^2 \int_{-\pi/2}^{\pi/2} \cos^2\theta\, d\theta$$

Next? We use the formula

$$\cos^2\theta = [1 - \cos(2\theta)]/2$$

$$\int \cos^2\theta \, d\theta = [\theta - \tfrac{1}{2}\sin(2\theta)]/2$$

to get

$$\text{area} = a^2 \left(\theta - \tfrac{1}{2}\sin(2\theta)\right)\Big|_{-\pi/2}^{\pi/2}$$

$$= a^2\left(\frac{\pi}{2} - 0 - \left(-\frac{\pi}{2} - 0\right)\right)$$

$$= \pi a^2 \qquad \text{Yup } \checkmark$$

- Polar Coordinates. Let $x = r\cos\theta$ & $y = r\sin\theta$. Then the area of a tiny piece, dA, satisfies the "formula"

$$dA = dx\,dy = r\,dr\,d\theta$$

Since the circle is easy to parametrize,

$$0 \leq r \leq a$$
$$0 \leq \theta \leq 2\pi,$$

we obtain

$$\text{area of circle} = \iint 1 \, r\,dr\,d\theta$$

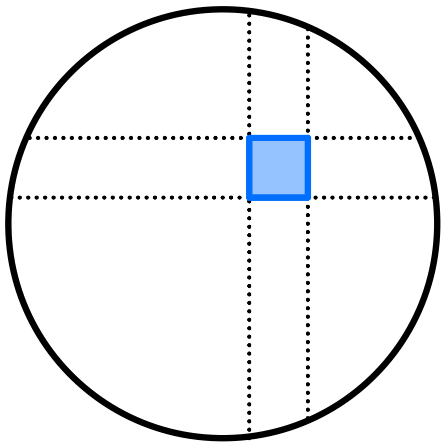
$$= \int_{\theta=0}^{\theta=2\pi} \left(\int_{r=0}^{r=a} r\,dr \right) d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} \frac{1}{2} a^2 \, d\theta$$

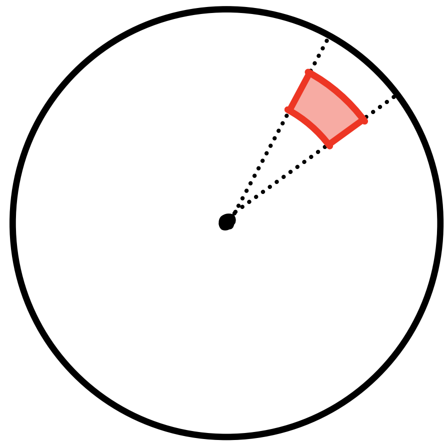
$$= 2\pi \frac{1}{2} a^2 = \pi a^2 \quad \checkmark$$

That was the right way to do it.

Lesson: Always use symmetry to simplify the computations.



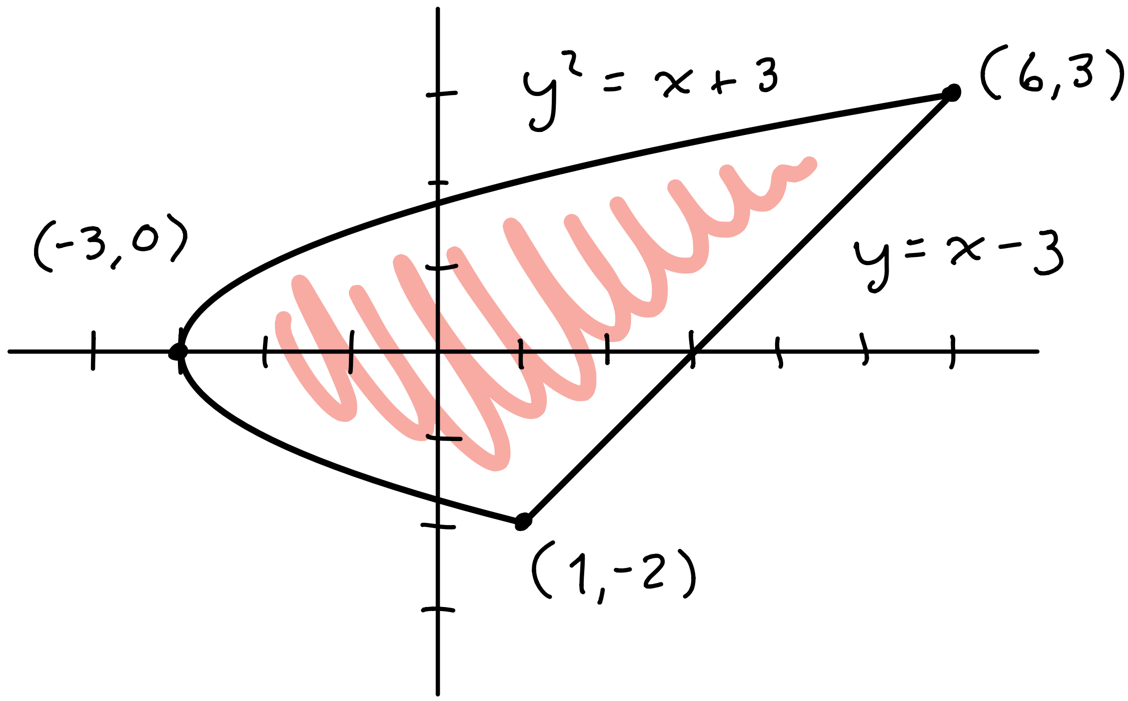
$dA = dx\,dy$



$dA = r\,dr\,d\theta$

Sadly, most problems don't have any symmetry so we have to use brute force.

Example: Consider the 2D region between the parabola $y^2 = x + 3$ and the line $y = x - 3$.



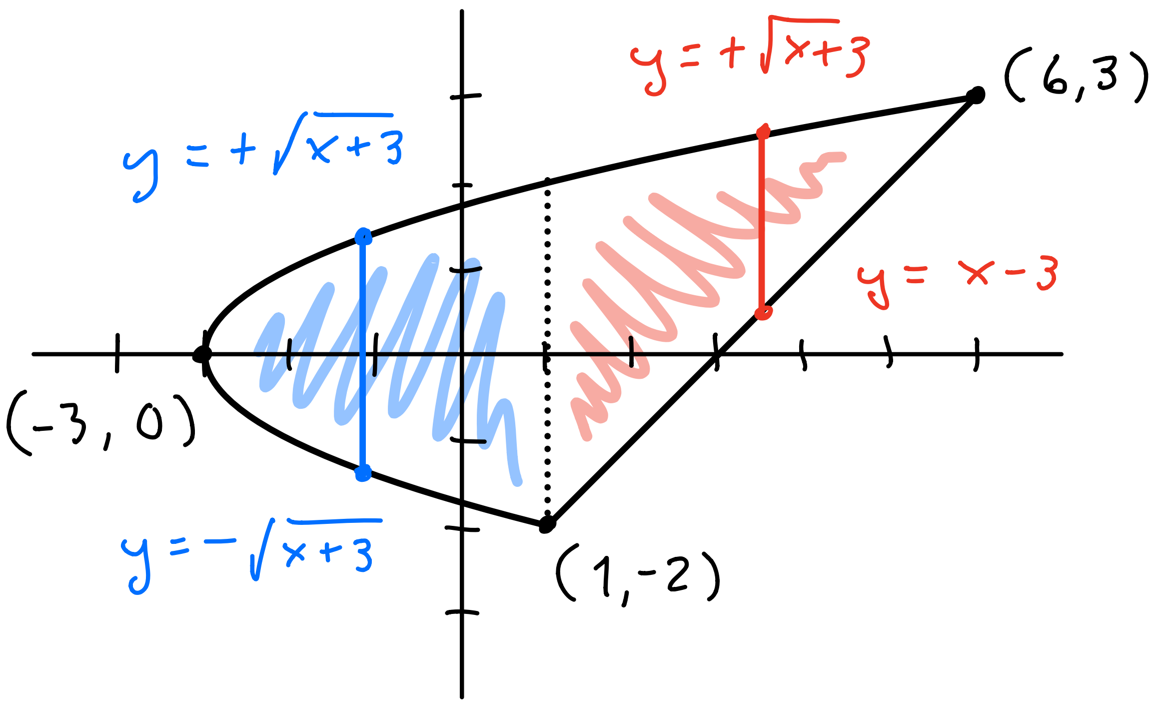
Compute the total mass if the density at (x,y) is

$$\rho(x,y) = 3x^2 + y^2.$$

The hard part is to choose a parametrization of the region.

Two options:

- Vertical slices. Fix x and find the limits of y.



We have to break the integral into two pieces:

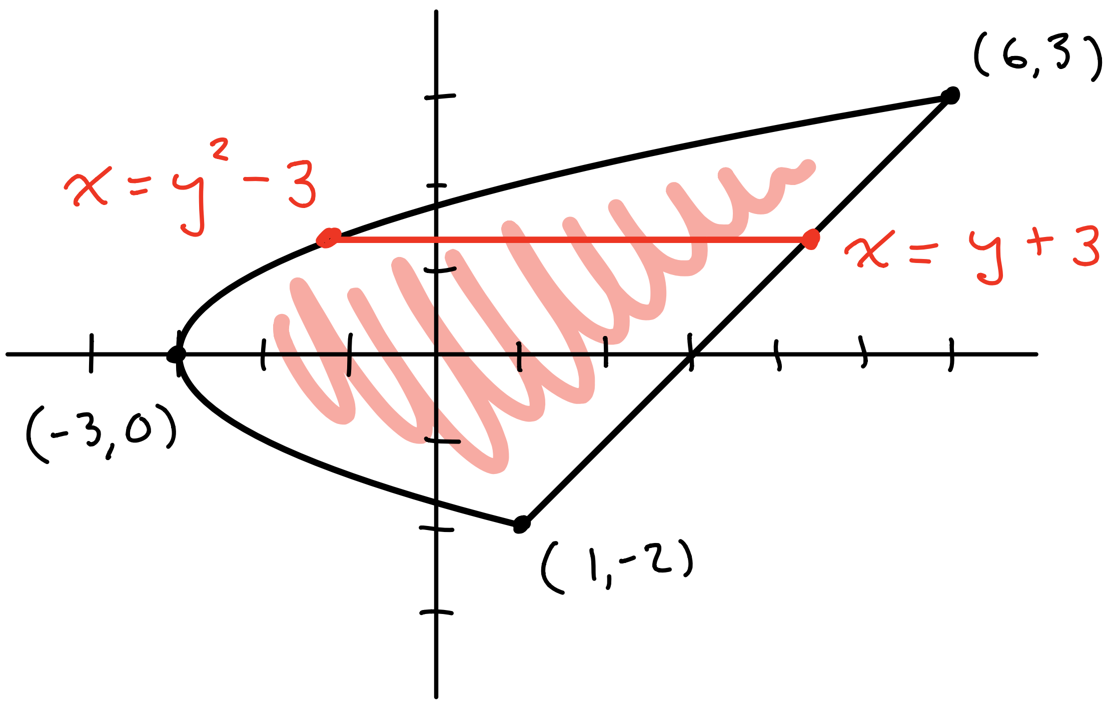
$$\text{mass} = \text{mass of left piece} + \text{mass of right piece}$$

$$= \int_{x=-3}^{x=+1} \left(\int_{y=-\sqrt{x+3}}^{y=+\sqrt{x+3}} (3x^2+y^2)\,dy \right) dx$$

$$+ \int_{x=1}^{x=6} \left(\int_{y=x-3}^{y=+\sqrt{x+3}} (3x^2+y^2)\,dy \right) dx$$

This looks hard so let's skip to the other method.

- Horizontal Slices. Fix y and find the limits on x.



This parametrization has two advantage over the other:

- We only have to compute one integral.
- There are no square roots.

!! ◡

$$\text{mass} = \int_{y=-2}^{y=+3} \left(\int_{x=y^2-3}^{x=y+3} (3x^2+y^2)\,dx \right) dy$$

$$= \int_{y=-2}^{y=+3} \left(\frac{3}{3}x^3 + y^2 x \right)\Bigg|_{x=y^2-3}^{x=y+3} dy$$

$\vdots$ skip

$$= \int_{-2}^{+3} \left(54 + 27y - 12y^2 + 2y^3 + 8y^4 - y^6\right) dy$$

$\vdots$ skip

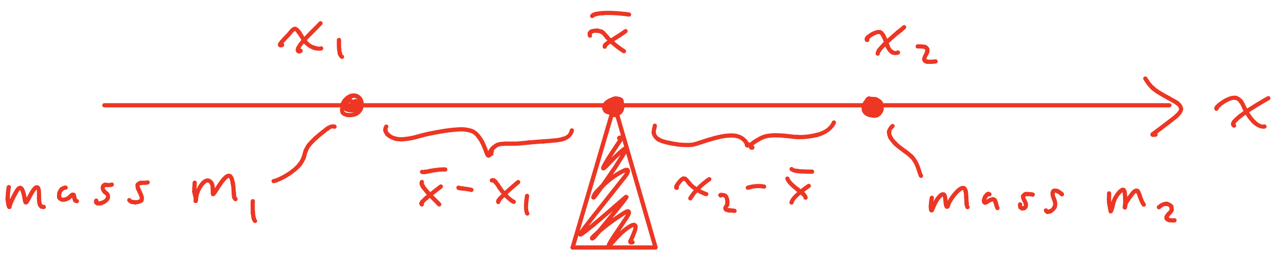
$$= \frac{2375}{7} \approx 339 \text{ units of mass.}$$

That could have been done by hand, but I used a computer.

What about the center of mass?

Where does it balance?

Archimedes' Law of the Lever:



If $\bar{x}$ is the balance point (the "center of mass") of two point masses on the x-axis then

$$m_1(\bar{x} - x_1) = m_2(x_2 - \bar{x})$$

$$\bar{x} = \frac{x_1 m_1 + x_2 m_2}{m_1 + m_2}$$

If we have n point masses $m_1, m_2, \ldots, m_n$ at positions $x_1, \ldots, x_n$

then this formula becomes

$$\bar{x} = \frac{x_1 m_1 + x_2 m_2 + \cdots + x_n m_n}{m_1 + m_2 + \cdots + m_n}$$

$$= \frac{\sum x_i m_i}{\sum m_i}$$

And for a continuous density function $\rho(x)$ on the x-axis, this becomes

$$\bar{x} = \frac{\int x \rho(x)\, dx}{\int \rho(x)\, dx}$$

In two dimensions: The center of mass has coordinates

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m} \right)$$

where

$$M_y = \iint x\,\rho(x,y)\,dA$$

$$M_x = \iint y\,\rho(x,y)\,dA$$

are the "moments about the x- and y-axes" and

$$m = \iint \rho(x,y)\,dx\,dy$$

is total mass. In our example, my computer gives

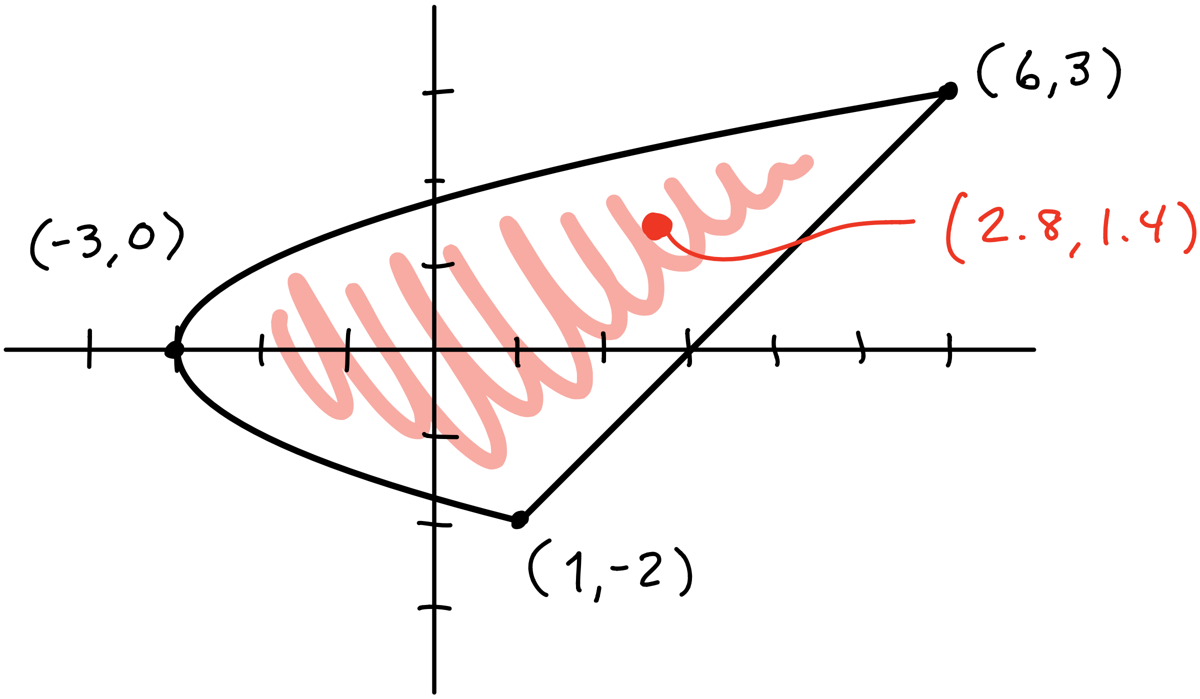
$$M_x = 11125/24$$

$$M_y = 39875/42,$$

so the center of mass is

$$(\bar{x},\bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m}\right) \approx (2.8, 1.4).$$

Picture :



The thin plate will balance on a pencil at the point $\approx (2.8, 1.4)$.