Quiz 5 Solutions:

1. Consider $f(x,y,z) = xy + z$.
Compute the integral of the gradient vector field $\vec{F} = \nabla f$ along the path $\vec{r}(t) = \langle t, t^2, t^3 \rangle$ from $t=0$ to $t=1$.

Easy Way (Fundamental Theorem of line integrals):

$$\int_0^1 \nabla f(\vec{r}(t)) \circ r'(t)\, dt$$

$$= f(\vec{r}(1)) - f(\vec{r}(0))$$

$$= f(1,1,1) - f(0,0,0)$$

$$= (1 \cdot 1 + 1) - (0 \cdot 0 + 0)$$

$$= 2.$$

Hard Way: First compute

$$\nabla f(x,y,z) = \langle y, x, 1 \rangle,$$

$$\nabla f(\vec{r}(t)) = \langle t^3, t, 1 \rangle, \quad \text{and}$$

$$\vec{r}\,'(t) = \langle 1, 2t, 3t^2 \rangle .$$

Then we have

$$\int_0^1 \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

$$= \int_0^1 \langle t^2, t, 1 \rangle \cdot \langle 1, 2t, 3t^2 \rangle \, dt$$

$$= \int_0^1 (t^2 + 2t^2 + 3t^2)\, dt$$

$$= \int_0^1 6t^2 \, dt$$

$$= 6\left(\frac{1}{3}t^3\right)\Big|_0^1 = 2 \quad \checkmark$$

2. Compute the circulation of

$$\vec{F}(x,y) = \left\langle -\frac{y}{2} + e^x,\ \frac{x}{2} - \ln(y) \right\rangle$$

around the circle $\vec{r}(t) = \langle \cos t, \sin t \rangle$ from $t = 0$ to $t = 2\pi$.

Easy Way (Green's Theorem):

First compute the curl

$$Q_x - P_y = \left(\frac{1}{2} + 0\right) - \left(-\frac{1}{2} + 0\right) = 1$$

If D is the interior of the unit circle C then Green's Theorem says

$$\oint_C \vec{F} \cdot \vec{T}\, ds = \iint_D \overset{1}{\cancel{\text{curl}(\vec{F})}}\, dA$$

$$= \iint_D dA$$

$$= \text{area of unit circle}$$

$$= \pi.$$

Hard Way: Since

$$\vec{F}(\vec{r}(t)) = \left\langle -\frac{\sin t}{2} + e^{\cos t}, \frac{\cos t}{2} - \ln(\sin t) \right\rangle$$

and

$$\vec{r}\,'(t) = \langle -\sin t, \cos t \rangle,$$

we have

$$\int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

$$= \int_0^{2\pi} \left[\frac{\sin^2 t}{2} - \sin t\, e^{\cos t} + \frac{\cos^2 t}{2} - \cos t \ln(\sin t) \right] dt$$

$\vdots$ hard to do by hand

$= \pi$ (via computer) ✓