Review for Quiz 5 :

- To integrate a scalar field f over a 1D oriented curve C :

$$\int_C f \underbrace{ds}_{\text{little piece of length}} = \int f(\vec{r}(t)) \underbrace{\|\vec{r}\,'(t)\| dt}_{\text{little piece of length}}$$

Application : The total area of a wall with base curve $\vec{r}(t) = \langle x(t), y(t) \rangle$ in the xy-plane and height $f(x,y)$ is

$$\int \overbrace{f(\vec{r}(t)) \|\vec{r}\,'(t)\| dt}^{\text{area of a skinny board}}$$

height of a skinny board

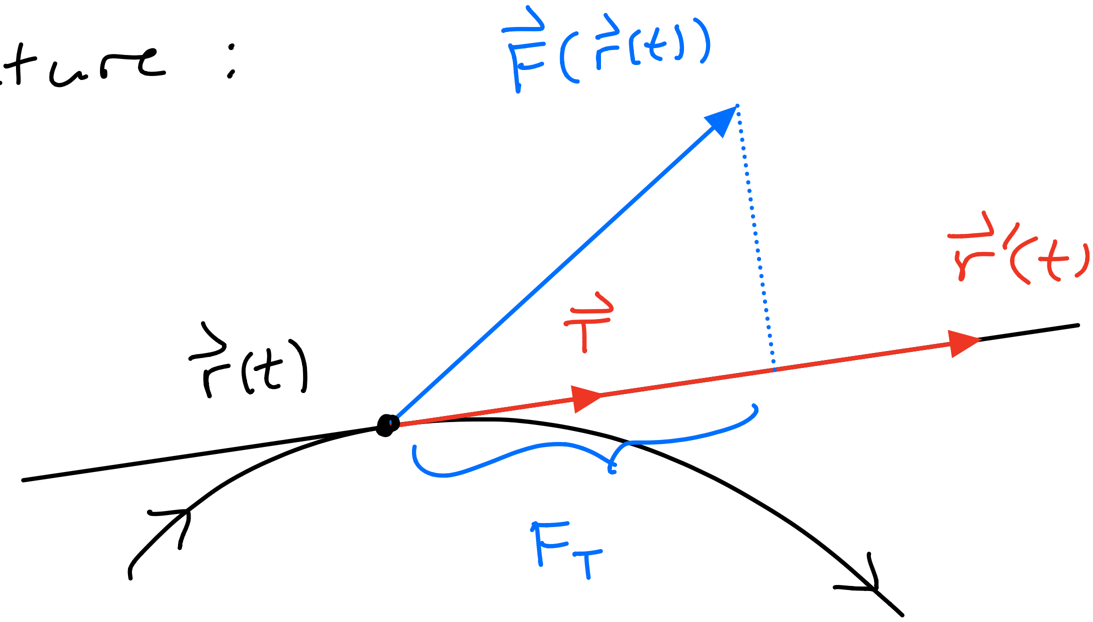
width of the skinny board.

$$= \int f(x(t), y(t)) \sqrt{x'(t)^2 + y'(t)^2}\, dt$$

- To integrate a vector field $\vec{F}$ over an oriented curve C:

$$\int_C \vec{F} \cdot \vec{T}\, ds = \int \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

Picture:



Consider the unit vector in the direction of your velocity:

$$\vec{T}(\vec{r}(t)) = \vec{r}\,'(t) \big/ \|\vec{r}\,'(t)\|$$

By HW 5.1, the component of $\vec{F}$ in the direction of $\vec{T}$ is just the dot product:

$$F_T = \vec{F} \circ \vec{T}$$

We integrate this scalar quantity over the curve:

$$\int_C \vec{F} \circ \vec{T} \, ds = \int_C F_T \, ds$$

- Conservative Vector Fields.

The following statements are equivalent:

- $\vec{F} = \nabla f$ for some scalar field
- $\mathrm{curl}(\vec{F}) = 0$ everywhere
- $\oint_{\text{loop}} \vec{F} \circ \vec{T} \, ds = 0$
- $\int_C \vec{F} \circ \vec{T} \, ds$ only depends on the endpoints of C; not the shape.

The equivalencies are described in terms of fundamental theorems.

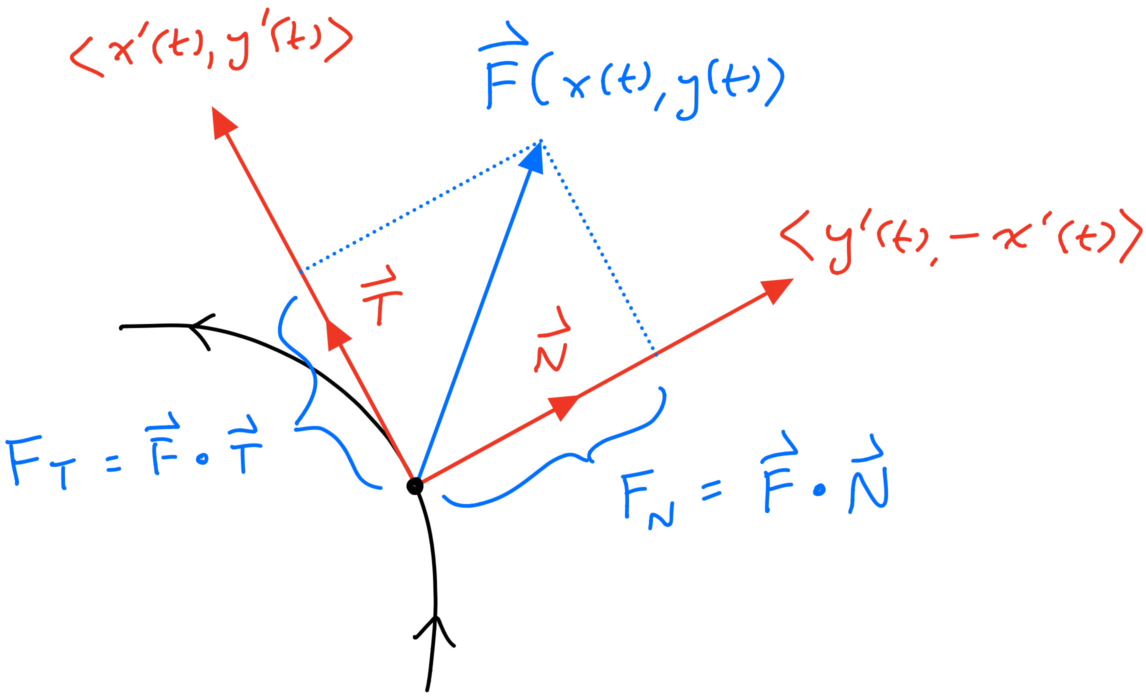
- Fund. Thm. of Line Integrals:

$$\int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt = f(\vec{r}(b)) - f(\vec{r}(a))$$

- Green's Theorem:

$$\iint_D \operatorname{curl}(\vec{F})\,dA = \oint_{\partial D} \vec{F} \cdot \vec{T}\,ds$$

- Flux Form of Green's Theorem:



For a path $\vec{r}(t) = \langle x(t), y(t) \rangle$ in 2D, we also have a <u>unit normal vector</u>

$$\vec{N}(\vec{r}(t)) = \frac{\langle y'(t), -x'(t) \rangle}{\| \langle y'(t), -x'(t) \rangle \|}$$

$$= \frac{1}{\| \vec{r}\,'(t) \|} \langle y'(t), -x'(t) \rangle$$

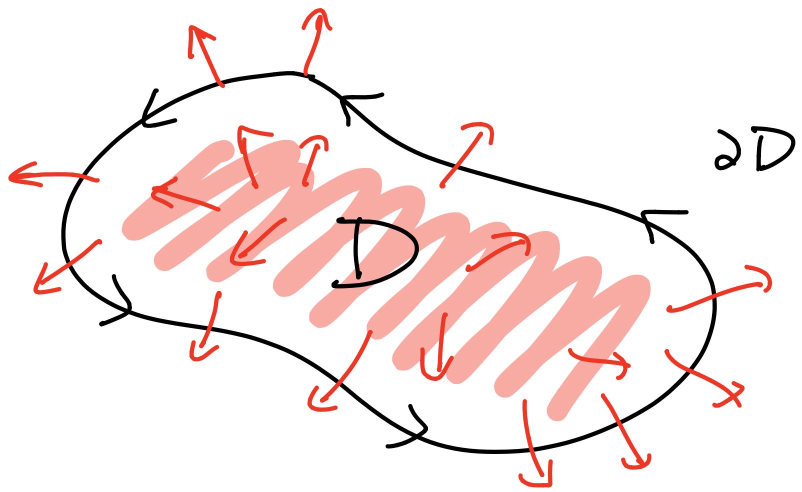
The "flux of $\vec{F}$ across the oriented curve" is the integral of the "normal component of $\vec{F}$":

$$\int_C \vec{F} \cdot \vec{N} \, ds = $$ how much does $\vec{F}$ point "to the right" of the curve?

Green's Theorem (Flux Form):

$$\iint_D \operatorname{div}(\vec{F}) \, dA = \oint_{\partial D} \vec{F} \cdot \vec{N} \, ds$$

Picture :



amount that $\vec{F}$ expands/contracts inside D = amount that $\vec{F}$ flows across the boundary ∂D

- Surface area of a parametrized 2D surface in 3D :

$$\vec{r} : \mathbb{R}^2 \to \mathbb{R}^3$$

$$\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$$

$$dS = \| \vec{r}_u \times \vec{r}_v \| \, du \, dv$$

= area of a tiny parallelogram on the surface at the point $\vec{r}(u,v)$.

So the total surface area is

$$\iint dS = \iint \|\vec{r}_u \times \vec{r}_v\| \, du \, dv$$

Example 6: Area of the surface $z = xy$ above the rectangle

$$0 \le x \le 1,$$
$$0 \le y \le 1.$$

If we choose the parametrization

$$\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$$

$$= \langle u, v, uv \rangle$$

$$\vec{r}_u = \langle 1, 0, v \rangle$$

$$\vec{r}_v = \langle 0, 1, u \rangle$$

$$\vec{r}_u \times \vec{r}_v = \langle -v, -u, 1 \rangle$$

$$\|\vec{r}_u \times \vec{r}_v\| = \sqrt{v^2 + u^2 + 1} \, du \, dv$$

then we can compute the area as follows:

$$\text{area} = \int_0^2 \left(\int_0^1 \sqrt{1+u^2+v^2}\, du \right) dv$$

$$\vdots \text{ computer}$$

$$\approx 3.18$$

[These kinds of integrals can <u>rarely</u> be done by hand!]

Picture: