Math 211 **Summer 2021**
Homework 5 **Drew Armstrong**

Problem 1. Projection. Let $\mathbf{F}, \mathbf{v}$ be two vectors in any dimensional space with $\|\mathbf{v}\| \neq 0$ and let $\hat{\mathbf{v}} = \mathbf{v}/\|\mathbf{v}\|$ be the unit vector in the direction of $\mathbf{v}$.

(a) The component of $\mathbf{F}$ in the direction of $\mathbf{v}$ has the form $F_{\mathbf{v}}\hat{\mathbf{v}}$ for some scalar $F_{\mathbf{v}}$. Show that we must have $F_{\mathbf{v}} = \mathbf{F} \bullet \hat{\mathbf{v}}$. [Hint: This is easier than is looks. By definition, the vector $\mathbf{F} - F_{\mathbf{v}}\hat{\mathbf{v}}$ is perpendicular to $\mathbf{v}$, so the dot product of these two vectors is zero.]

(b) Draw a picture of the situation in part (a).

Problem 2. Area of a Wall. Find the area of the wall below the surface $z = x^2 + y + 1$ whose base is the ellipse $(x/2)^2 + y^2 = 1$ in the xy-plane. This integral is not solvable by hand, so use a computer to get a numerical approximation. [Hint: Use $\mathbf{r}(t) = \langle 2\cos t, \sin t\rangle$.]

Problem 3. Surface Area of a Parabolic Dome. Consider the parabolic dome between the xy-plane and the parabolic surface $z = 1 - x^2 - y^2$.

(a) Find a parametrization for the top surface of the dome:

$$\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle.$$

(b) Use the formula $\iint dS = \iint \|\mathbf{r}_u \times \mathbf{r}_v\|\, du\, dv$ to compute the area of the top surface of the dome. [Hint: Polar coordinates are easier.]

Problem 4. Conservation of Energy. If a particle of mass m follows a trajectory $\mathbf{r}(t)$ then we define its kinetic energy at time t by $\mathrm{KE}(t) = m\|\mathbf{v}(t)\|^2/2 = m\|\mathbf{r}'(t)\|^2/2$.

(a) If $\mathbf{F}$ is any force field acting on the particle then Newton's Second Law says that $\mathbf{F}(\mathbf{r}(t)) = m\mathbf{r}''(t)$. Use this to prove that

$$\int_a^b \mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{r}'(t)\, dt = \mathrm{KE}(b) - \mathrm{KE}(a).$$

[Hint: Write $\mathrm{KE}(t) = m\mathbf{r}'(t) \bullet \mathbf{r}'(t)/2$ and differentiate this using the product rule for dot products. Then use the Calculus I version of the Fundamental Theorem.]

(b) Furthermore, suppose that $\mathbf{F} = -\nabla f$ is a conservative vector field, so the particle also has a potential energy $\mathrm{PE}(t) = f(\mathbf{r}(t))$ at time t. Use part (a) and the Fundamental Theorem of Line Integrals to prove that

$$\mathrm{KE}(a) + \mathrm{PE}(a) = \mathrm{KE}(b) + \mathrm{PE}(b).$$

Problem 5. Gravity in Two Dimensions (Green's Theorem). Consider the vector field $\mathbf{F}(x, y) = \frac{-1}{x^2+y^2}\langle x, y\rangle$, which is not defined at the origin $(0, 0)$.

(a) Show that $\mathrm{curl}(\mathbf{F})(x, y) = 0$ when $(x, y) \neq (0, 0)$, so the circulation of $\mathbf{F}$ around any loop not containing $(0, 0)$ is zero.

(b) Show that the circulation around any loop containing $(0, 0)$ is also zero. (It doesn't need to be, but it is.) [Hint: Use $\mathbf{r}(t) = \langle \cos t, \sin t\rangle$.]

(c) Show that $\mathrm{div}(\mathbf{F})(x, y) = 0$ when $(x, y) \neq (0, 0)$, so the flux of $\mathbf{F}$ across any loop not containing $(0, 0)$ is zero.

(d) However, the flux of $\mathbf{F}$ across a loop containing $(0, 0)$ is not zero. What is it?