1. Let $f$ be a function of $x$ & $y$, where $x = s^2 + t^2$ & $y = 2st$.

a) Express $f_s$ & $f_t$ in terms of $s, t, f_x$ & $f_y$.

First we note that

$$dx/ds = 2s$$
$$dx/dt = 2t$$
$$dy/ds = 2t$$
$$dy/dt = 2s.$$

Then by the chain rule we have

$$\frac{df}{ds} = \frac{df}{dx}\frac{dx}{ds} + \frac{df}{dy}\frac{dy}{ds}$$

$$f_s = f_x \cdot 2s + f_y \cdot 2t$$

and

$$\frac{df}{dt} = \frac{df}{dx}\frac{dx}{dt} + \frac{df}{dy}\frac{dy}{dt}$$

$$f_t = f_x \cdot 2t + f_y \cdot 2s.$$

b) Express $f_{ss}$ in terms of

$s, t, f_{xx}, f_{yy}$ & $f_{xy}$ ($= f_{yx}$)

[ For nice functions (e.g. the type that come up in applications) we always have $f_{xy} = f_{yx}$. That is,

$$\frac{d}{dy}\left(\frac{df}{dx}\right) = \frac{d}{dx}\left(\frac{df}{dy}\right). \quad ]$$

First use the chain rule to compute

$$\frac{df_x}{ds} = \frac{df_x}{dx}\frac{dx}{ds} + \frac{df_x}{dy}\frac{dy}{ds}$$

$$f_{xs} = f_{xx}\cdot 2s + f_{xy}\cdot 2t$$

and

$$\frac{df_y}{ds} = \frac{df_y}{dx}\frac{dx}{ds} + \frac{df_y}{dy}\frac{dy}{ds}$$

$$f_{ys} = f_{yx}\cdot 2s + f_{yy}\cdot 2t.$$

[ Remark: $f_x$ & $f_y$ are functions of $x$ & $y$. ]

Then we use part (a) and the product rule to obtain

$$f_{ss} = \frac{d}{ds} f_s$$

$$= \frac{d}{ds}\left(2s f_x + 2t f_y\right)$$

$$= 2\left(\overset{1}{\cancel{\frac{ds}{ds}}} f_x + s\frac{df_x}{ds}\right)$$

$$+ 2\left(\overset{0}{\cancel{\frac{dt}{ds}}} f_y + t\frac{df_y}{ds}\right)$$

(assume $t$ is not a function of $s$)

$$= 2f_x + 2s f_{xs} + 2t f_{ys}$$

$$= 2f_x + 2s\left(2s f_{xx} + 2t f_{xy}\right)$$

$$+ 2t\left(2s f_{yx} + 2t f_{yy}\right)$$

$$\boxed{= 2f_x + 4s^2 f_{xx} + 4t^2 f_{yy} + 8st f_{xy}}$$

One could also compute formulas for $f_{tt}$ & $f_{st}$ $(= f_{ts})$.

## 2. Integration over a Rectangle.

Find the volume between the surface $z = (x+y)^2 = x^2 + 2xy + y^2$ and the rectangle $a_1 \le x \le a_2$ & $b_1 \le y \le b_2$.

$$\text{volume} = \iint \text{volume of skinny column}$$

$$= \iint \underbrace{f(x,y)}_{\text{height}} \underbrace{dx\,dy}_{\text{area of base}}$$

$$= \int_{b_1}^{b_2} \left( \int_{a_1}^{a_2} (x^2 + 2xy + y^2)\, dx \right) dy$$

$$= \int_{b_1}^{b_2} \left( \frac{1}{3}x^3 + x^2 y + x y^2 \right)\Big|_{x=a_1}^{x=a_2} dy$$

$$= \int_{b_1}^{b_2} \frac{1}{3}(a_2^3 - a_1^3) + (a_2^2 - a_1^2)y + (a_2 - a_1)y^2 \, dy$$

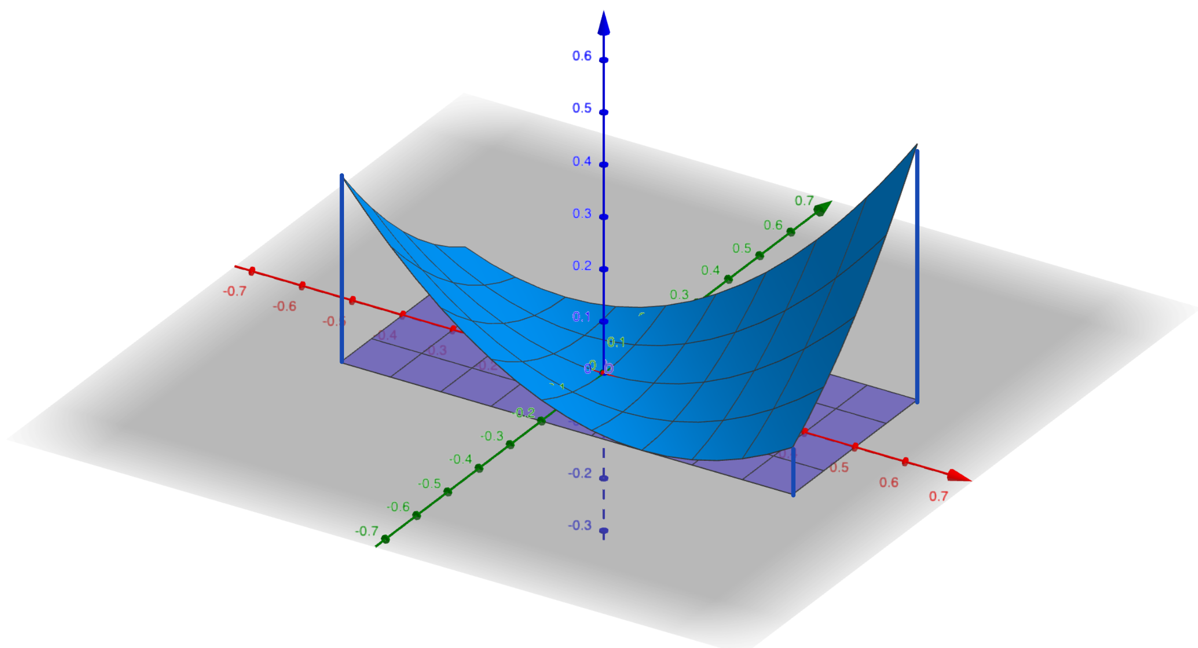
$$= \left( \frac{1}{3}(a_2^3 - a_1^3)y + \frac{1}{2}(a_2^2 - a_1^2)y^2 + \frac{1}{3}(a_2 - a_1)y^3 \right)\Big|_{b_1}^{b_2}$$

$$= \frac{1}{3}(a_2^3 - a_1^3)(b_2 - b_1) + \frac{1}{2}(a_2^2 - a_1^2)(b_2^2 - b_1^2) + \frac{1}{3}(a_2 - a_1)(b_2^3 - b_1^3)$$

[We could factor out $(a_2 - a_1)(b_2 - b_1)$ but that doesn't make it look nicer.]

Here is a picture of the bottom and top of the region when

$$(a_1, a_2, b_1, b_2) = (-0.4, 0.5, -0.2, 0.2)$$

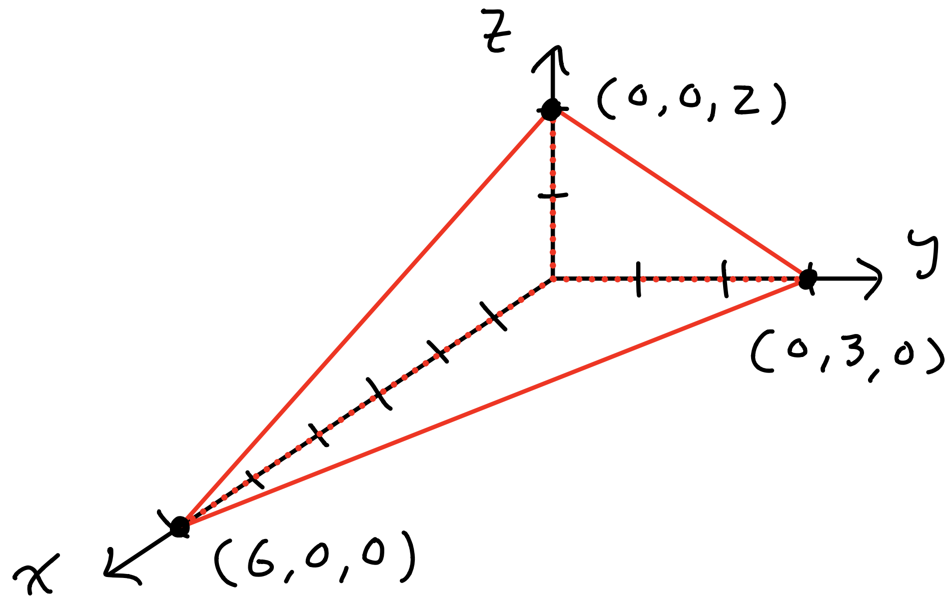



The volume of this region is 0.03

3. Integration over a Tetrahedron.

Consider the region in $\mathbb{R}^3$ where

$$x, y, z \geq 0 \quad \& \quad x + 2y + 3z \leq 6 :$$




Suppose this region has mass density

$$\rho(x,y,z) = 1 + x.$$

a) Compute the total mass.

I claim that we can parametrize the tetrahedron as follows:

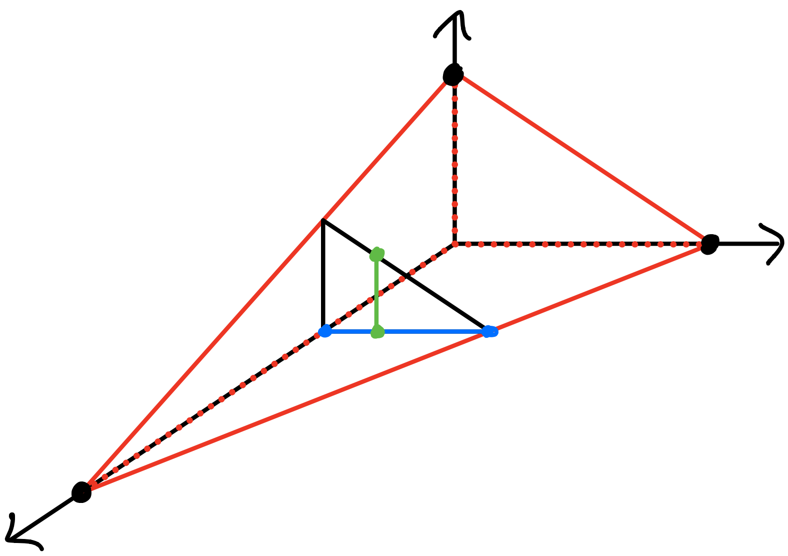
$$\begin{cases} 0 \leq x \leq 6 \\ 0 \leq y \leq 3 - 3x/6 \\ 0 \leq z \leq 2 - 2x/6 - 2y/3 \end{cases}$$

We could get this from the formula in the lecture notes with

$$(a, b, c) = (6, 3, 2).$$

Or we can do it from scratch.

- Fix some $0 \leq x \leq 6$ and consider the triangular slice:



- The bottom of this triangular slice has $z = 0$, so $y$ satisfies

$$x + 2y + 0 \leq 6$$
$$2y \leq 6 - x$$
$$y \leq (6 - x)/2,$$ as claimed ✓

- Then for fixed $x$ & $y$, the value of $z$ must satisfy

$$x + 2y + 3z \leq 6$$
$$3z \leq 6 - 2y - x$$
$$z \leq (6 - 2y - x)/3$$ , as claimed ✓

Now we use this parametrization to compute the mass:

$$\text{mass} = \iiint \text{density} \cdot dV$$

$$= \iiint \rho(x, y, z)\, dx\, dy\, dz$$

$$= \iint \left( \int_0^{2 - 2y/3 - x/3} (1+x)\, dz \right) dy\, dx$$

$$= \iint (1+x)\left(2 - \frac{2}{3}y - \frac{1}{3}x\right) dy\, dx$$

$$= \int \left( \int_0^{3 - x/2} \left[ (1+x)\left(2 - \frac{1}{3}x\right) - \frac{2}{3}(1+x)y \right] dy \right) dx$$

$$= \int_0^6 \Big[ (1+x)(2-\tfrac{1}{3}x)(3-\tfrac{x}{2}) - \frac{2}{3}(1+x)\frac{1}{2}(3-\frac{x}{2})^2 \Big] dx$$

⋮ computer

$$= \int_0^6 \left( 3 + 2x - \frac{11}{12}x^2 + \frac{1}{12}x^3 \right) dx$$

$$= 3\cdot 6 + \frac{1}{2}2\cdot 6^2 - \frac{1}{3}\cdot\frac{11}{12}6^3 + \frac{1}{4}\frac{1}{12}6^4$$

$= 15$ (units of mass)

b) Part (a) was a lot of work. This time I'll do everything on the computer, using the same parametrization of the domain.

$$M_{yz} = \iiint x(1+x)\,dV = \cdots = 153/5$$

$$M_{xz} = \iiint y(1+x)\,dV = \cdots = 99/10$$

$$M_{xy} = \iiint z(1+x)\,dV = \cdots = 33/5.$$

c) So the center of mass is

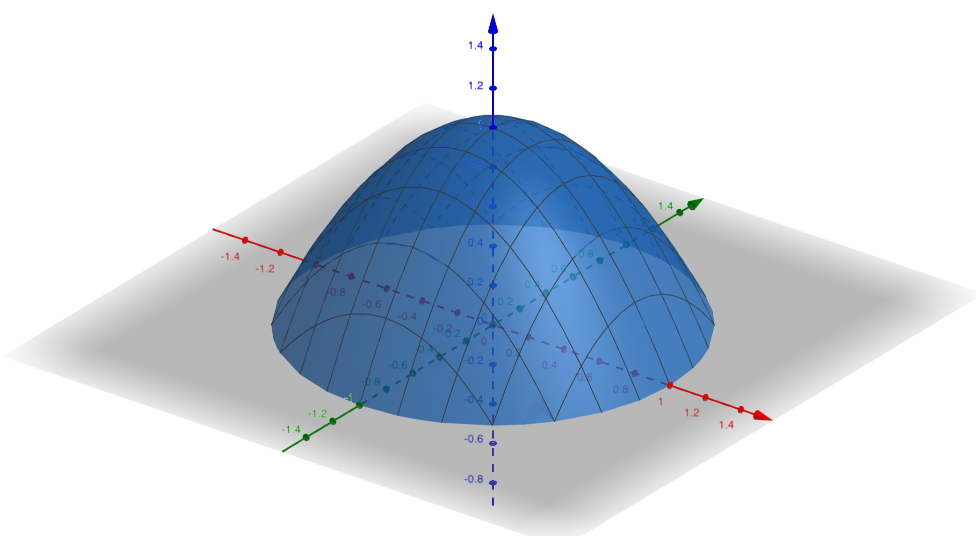
$$(\bar{x}, \bar{y}, \bar{z}) = \left( \frac{M_{yz}}{m}, \frac{M_{xz}}{m}, \frac{M_{xy}}{m} \right)$$

$$= \left( \frac{153/5}{15}, \frac{99/10}{15}, \frac{33/5}{15} \right)$$

$$= (2.04, 0.66, 0.44).$$

4. Cylindrical Coordinates.

Consider the parabolic dome between the unit disk $x^2+y^2 \le 1$ and the surface $z = 1-x^2-y^2$:



a) We will compute the mass using cylindrical coordinates. Note that the dome is parametrized by

$$0 \leq r \leq 1$$
$$0 \leq \theta \leq 2\pi$$
$$0 \leq z \leq 1-x^2-y^2 = 1-r^2,$$

so we get (density is 1)

$$\text{mass} = \iiint 1 \, dV$$

$$= \iiint r \, dr \, d\theta \, dz$$

$$= \iint \left( \int_0^{1-r^2} r \, dz \right) dr \, d\theta$$

$$= \iint r(1-r^2) \, dr \, d\theta$$

$$= \int \left( \int_0^1 (r - r^3) \, dr \right) d\theta$$

$$= \int \left( \frac{1}{2} r^2 - \frac{1}{4} r^4 \right)_0^1 \, d\theta$$

$$= \int_0^{2\pi} \frac{1}{4} \, d\theta$$

$$= 2\pi \cdot \frac{1}{4} = \pi/2 .$$

[We already did this computation in the lecture.]

b) The moment about the xy-plane is

$$M_{xy} = \iiint z \, dV$$

$$= \iiint z \, r \, dr \, d\theta \, dz$$

$$= \iint \left( \int_0^{1-r^2} z r \, dz \right) dr \, d\theta$$

$$= \iint \left( \frac{1}{2} z^2 r \right)_0^{1-r^2} dr \, d\theta$$

$$= \int \left( \int_0^1 \frac{1}{2}(1-r^2)^2 r \, dr \right) d\theta$$

$$= \int \left( \int_0^1 \frac{1}{2}(r - 2r^3 + r^5) \, dr \right) d\theta$$

$$= \int_0^{2\pi} \frac{1}{2} \left( \frac{1}{2} r^2 - \frac{2}{4} r^4 + \frac{1}{6} r^6 \right)_0^1 d\theta$$

$$= \int_0^{2\pi} \frac{1}{12} \, d\theta$$

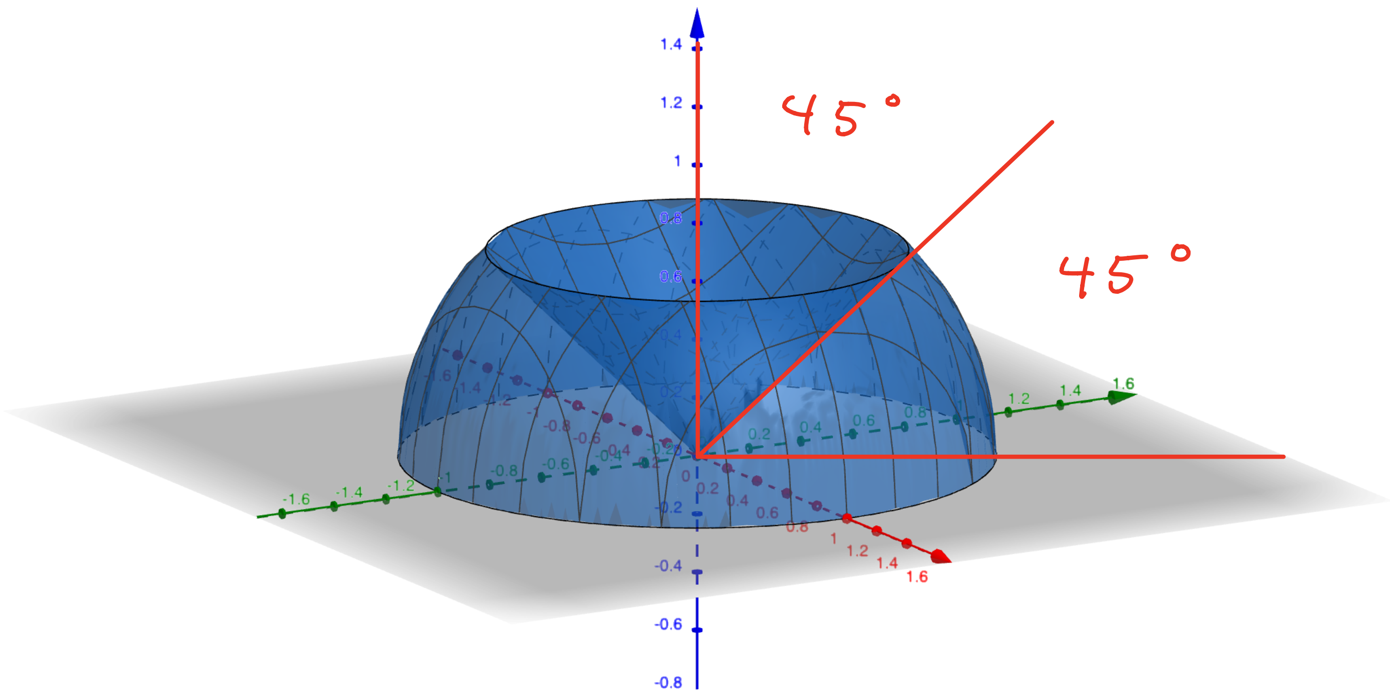
$$= 2\pi \cdot \frac{1}{12} = \pi/6 .$$

c) By symmetry we have $M_{xz} = M_{yz} = 0$, so the center of mass is

$$(\bar{x}, \bar{y}, \bar{z}) = \left( \frac{M_{yz}}{m}, \frac{M_{xz}}{m}, \frac{M_{xy}}{m} \right)$$

$$= \left( 0, 0, \frac{\pi/6}{\pi/2} \right) = \left( 0, 0, \frac{1}{3} \right).$$

6. We will compute the center of mass of the solid region above the $xy$-plane, below the cone $z = \sqrt{x^2+y^2}$ and inside the sphere $x^2+y^2+z^2 \leq 1$:




The region can be parametrized in spherical coordinates by

$$0 \leq \rho \leq 1$$
$$0 \leq \theta \leq 2\pi$$
$$\pi/4 \leq \varphi \leq \pi/2$$

a) The mass (or volume; we assume density is 1) is

$$m = \iiint dV$$

integrand is "separable"

$$= \iiint \rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi$$

$$= \int_0^{2\pi} d\theta \int_0^1 \rho^2 \, d\rho \int_{\pi/4}^{\pi/2} \sin\varphi \, d\varphi$$

$$= 2\pi \cdot \frac{1}{3} \cdot \left(-\cos\frac{\pi}{2} + \cos\frac{\pi}{4}\right)$$

$$= 2\pi \cdot \frac{1}{3} \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{3}\pi$$

b) The moment about the xy-plane is

$$M_{xy} = \iiint z \, dV$$

$$= \iiint \rho\cos\varphi \, \rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi$$

still "separable"

$$= \int_0^{2\pi} d\theta \int_0^1 \rho^3 \, d\rho \int_{\pi/4}^{\pi/2} \sin\varphi \cos\varphi \, d\varphi$$

$$= 2\pi \cdot \frac{1}{4} \int_{\pi/4}^{\pi/2} \sin\varphi \cos\varphi \, d\varphi$$

[ Now we use a trig identity:

$$2 \sin\varphi \cos\varphi = \sin(2\varphi)$$

$$\sin\varphi \cos\varphi = \frac{1}{2} \sin(2\varphi)$$ ]

$$= \cancel{2}\pi \cdot \frac{1}{4} \cdot \frac{1}{\cancel{2}} \int_{\pi/4}^{\pi/2} \sin(2\varphi) \, d\varphi$$

$$= \frac{\pi}{4} \left( -\frac{1}{2} \cos(2\varphi) \right)_{\pi/4}^{\pi/2}$$

$$= \frac{\pi}{4} \left( -\frac{1}{2} \overset{-1}{\cancel{\cos(\pi)}} + \frac{1}{2} \overset{0}{\cancel{\cos(\pi/2)}} \right)$$

$$= \pi/8$$

c) By symmetry we have $M_{xz} = M_{yz} = 0$,

so the center of mass is

$$(\bar{x}, \bar{y}, \bar{z}) = \left( \frac{M_{yz}}{m}, \frac{M_{xz}}{m}, \frac{M_{xy}}{m} \right)$$

$$= \left( 0, 0, \frac{\pi/8}{\pi\sqrt{2}/3} \right)$$

$$= \left( 0, 0, 3\sqrt{2}/16 \right)$$

$$\approx (0, 0, 0.265)$$

That seems reasonable.

Remark: If you throw this object in the air, its center of mass will trace out a parabolic curve.