1. Tangent Planes.

a) Find the tangent plane to the level surface of $f(x,y,z) = x^2 + y^2 - \frac{z}{5}$ through the point $(1,2,1)$.

The gradient vector is

$$\nabla f = \langle 2x, 2y, -\tfrac{1}{5} \rangle$$

$$\nabla f(1,2,1) = \langle 2, 4, -1/5 \rangle$$

So the equation of the tangent plane is

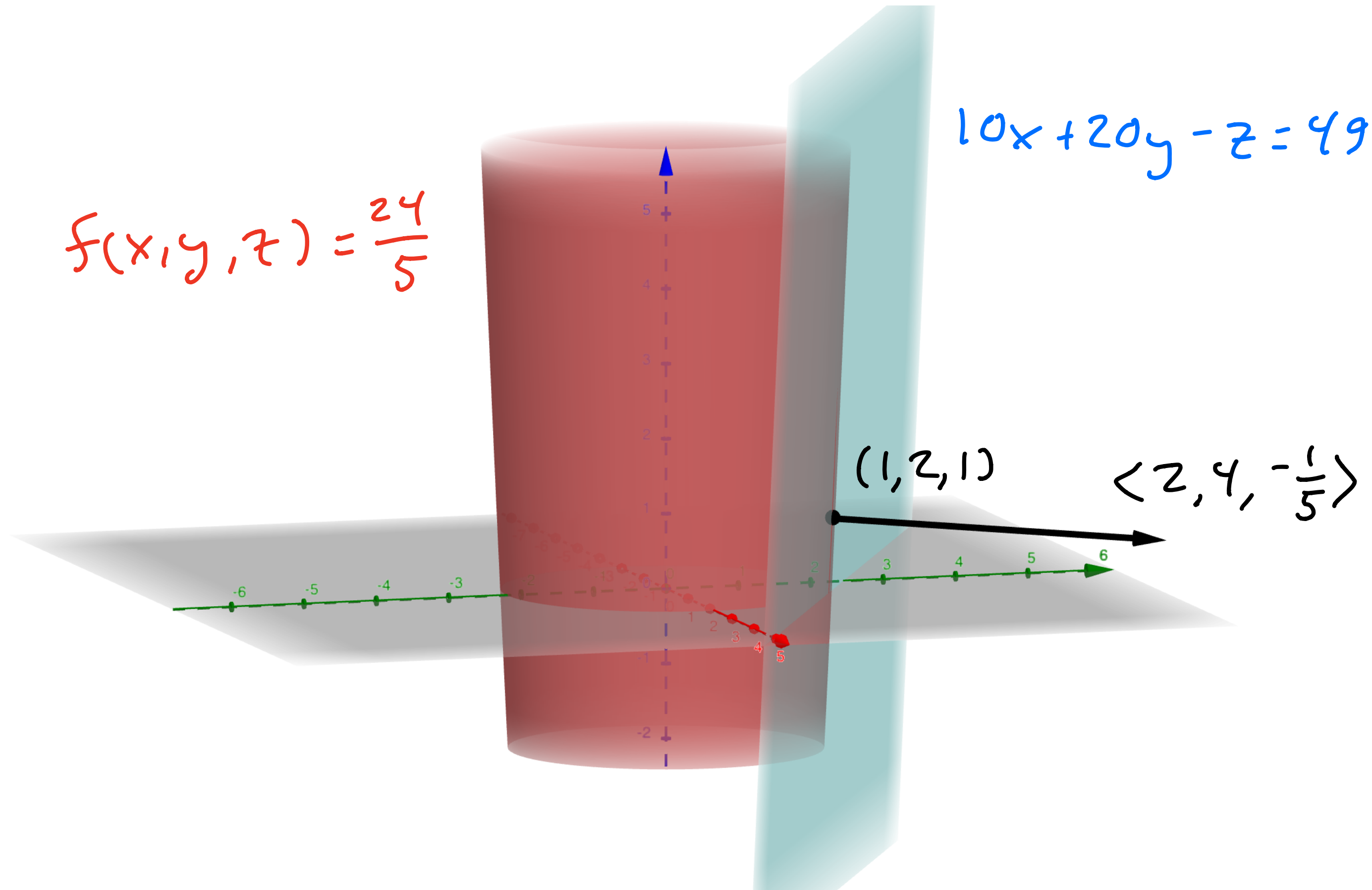
$$\langle 2, 4, -\tfrac{1}{5} \rangle \cdot \langle x-1, y-2, z-1 \rangle = 0$$

$$2(x-1) + 4(y-2) - \tfrac{1}{5}(z-1) = 0$$

$$2x + 4y - \tfrac{1}{5}z = 2 + 8 - \tfrac{1}{5} = \frac{49}{5}$$

$$10x + 20y - z = 49$$

Here is a picture:



b) Find the tangent plane to the level surface of $f(x,y,z) = x^2+y^2+z^2$ through the point $(1,-3,2)$.

The gradient vector is

$$\nabla f = \langle 2x, 2y, 2z \rangle$$

$$\nabla f(1,-3,2) = \langle 2,-6,4 \rangle$$

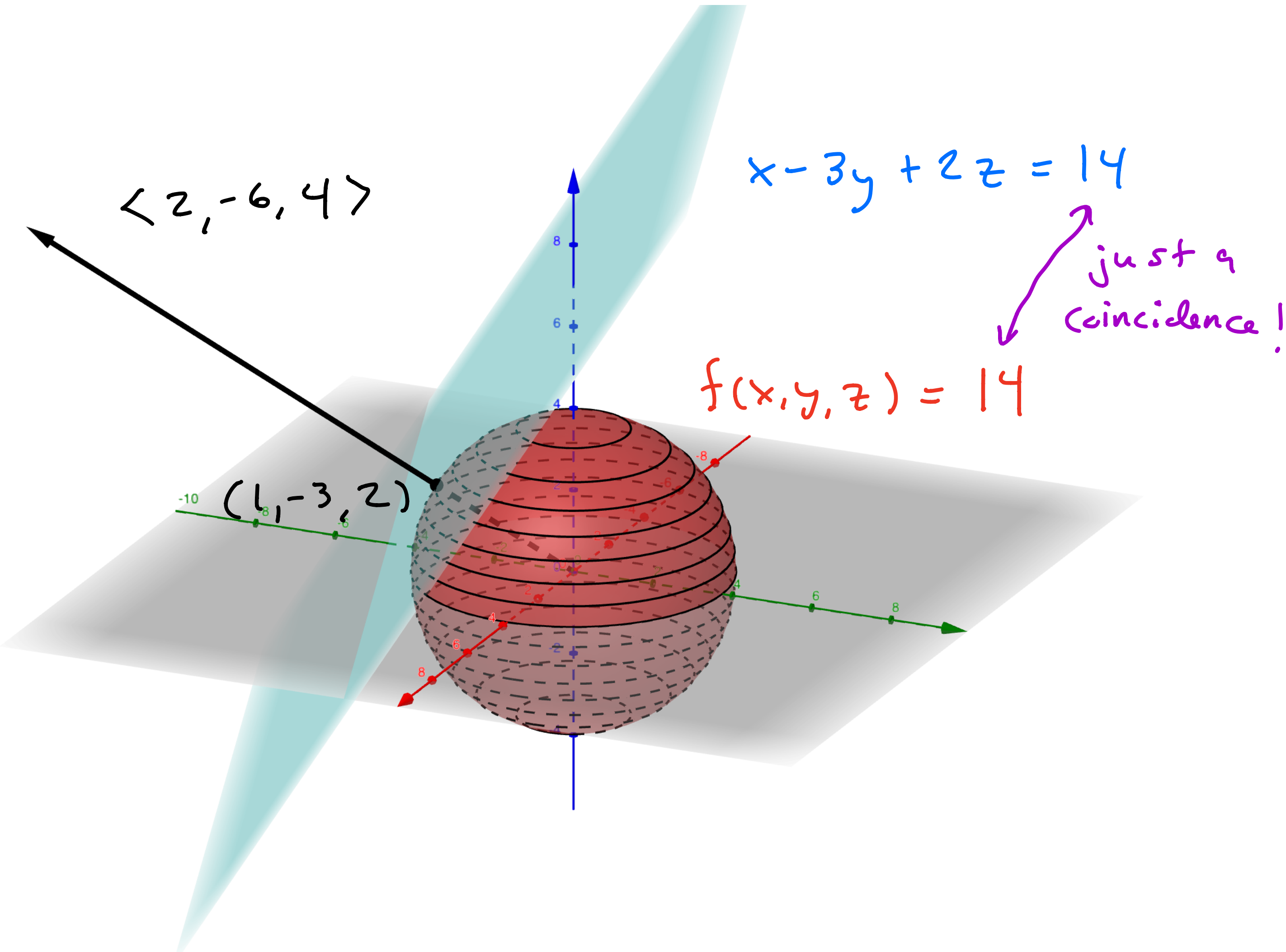
So the equation of the tang. plane is

$$2(x-1) - 6(x+3) + 4(z-2) = 0$$

$$2x - 6y + 4z = 2 + 18 + 8 = 28$$

$$x - 3y + 2z = 14$$

Here is a picture:



c) Find the tangent plane to the level surface of $f(x,y,z) = \left(\frac{x}{2}\right)^2 + \left(\frac{y}{2}\right)^2 + \left(\frac{z}{3}\right)^2$ through the point $(2, 2, 3)$.

The gradient vector is

$$\nabla f = \left\langle \tfrac{1}{2}x, \tfrac{1}{2}y, \tfrac{2}{9}z \right\rangle$$

$$\nabla f(2,2,3) = \left\langle 1, 1, \tfrac{2}{3} \right\rangle$$

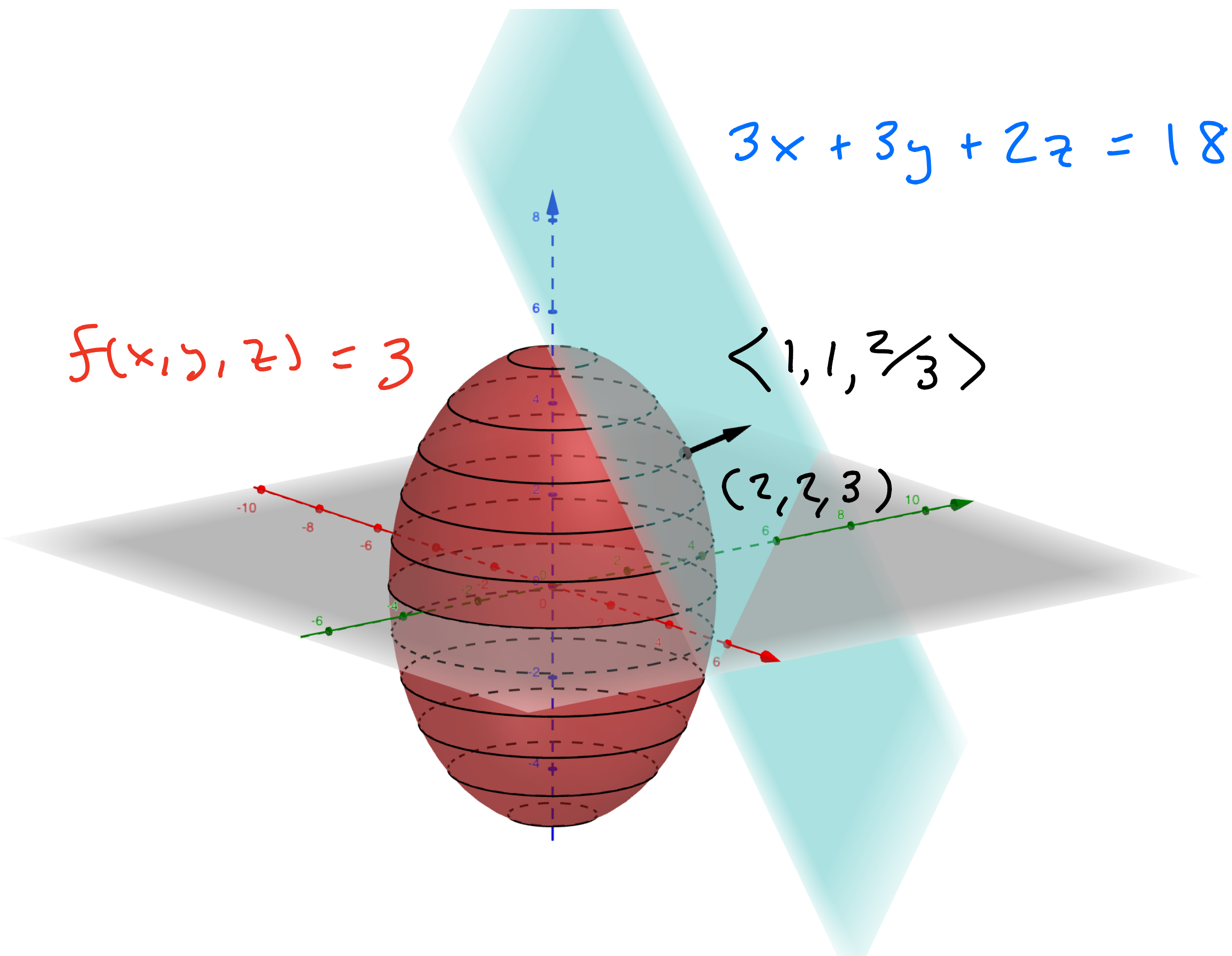
So the eqn of the tangent plane is

$$1(x-2) + 1(y-2) + \tfrac{2}{3}(z-3) = 0$$

$$x + y + \tfrac{2}{3}z = 2 + 2 + 2 = 6$$

$$3x + 3y + 2z = 18$$

Here is a picture:



2. Consider a rectangular box with length/width/heigt l, w, h.

If the box has an open top then the surface area is

$$A = lw + 2lh + 2wh$$

If the dimensions are measured as

$$l = 10 \pm 0.03$$
$$w = 6 \pm 0.02 \qquad \text{(inches)}$$
$$h = 3 \pm 0.01,$$

estimate the uncertainty in A.

We use the chain rule in "differential form":

$$dA = \frac{dA}{dl} dl + \frac{dA}{dw} dw + \frac{dA}{dh} dh$$

$$= (w + 2h) dl + (l + 2h) dw + (2l + 2w) dh$$

For finite differences this becomes

$$\Delta A \approx (w+2h)\,\Delta l$$
$$+ (l+2h)\,\Delta w$$
$$+ (2l+2w)\,\Delta h$$
$$= 12(0.03) + 16(0.02) + 32(0.01)$$
$$= 1.00 \text{ inches}^2$$

Thus we can say that

$$A \approx 156 \pm 1.00 \text{ inches}^2$$

3. Consider the scalar field

$$f(x,y) = 1 + 4xy - x^4 - y^4$$

Find the critical points & determine whether each is a local max/min or a saddle point.

First we compute the derivatives:

$$f_x = 4y - 4x^3$$
$$f_{xx} = -12x^2$$
$$f_{xy} = 4$$
$$f_y = 4x - 4y^3$$
$$f_{yx} = 4$$
$$f_{yy} = -12y^2.$$

The gradient is

$$\nabla f = \langle f_x, f_y \rangle = \langle 4y - 4x^3, 4x - 4y^3 \rangle$$

To find the critical points (a,b) we set

$$\nabla f(a,b) = \langle 0, 0 \rangle$$
$$\langle 4b - 4a^3, 4a - 4b^3 \rangle = \langle 0, 0 \rangle$$

which implies that

$$4b - 4a^3 = 0 \quad \& \quad 4a - 4b^3 = 0$$
$$b = a^3 \qquad\qquad a = b^3$$

Taking the quotient of these equations gives

$$\frac{b}{a} = \frac{a^3}{b^3} = \left(\frac{a}{b}\right)^3$$

$$\left(\frac{b}{a}\right)^4 = 1$$

$$\frac{b}{a} = \pm 1$$

But $b = a^3$ tells us that a & b have the same sign, so we must have

$$\frac{b}{a} = +1 \longrightarrow a = b.$$

Finally, the equation

$$b = a^3$$

$$a = a^3$$

$$a(1-a^2) = 0$$

$$a(1-a)(1+a) = 0$$

tells us that $a = 0, +1, -1$.

So the critical points are

$$(a,b) = (0,0), (1,1), (-1,-1).$$

Next we consider the 2×2 matrix of second derivatives (the "Hessian matrix":

$$Hf = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$$

$$= \begin{pmatrix} -12x^2 & 4 \\ 4 & -12y^2 \end{pmatrix}$$

$$\det(Hf) = 144x^2y^2 - 16$$

- The critical point $(0,0)$ has

$$\det(HF)(0,0) = -16 < 0$$

so it is a saddle point.

- The critical points $(1,1), (-1,-1)$

have $\det(Hf)(1,1) = 144-16 > 0$

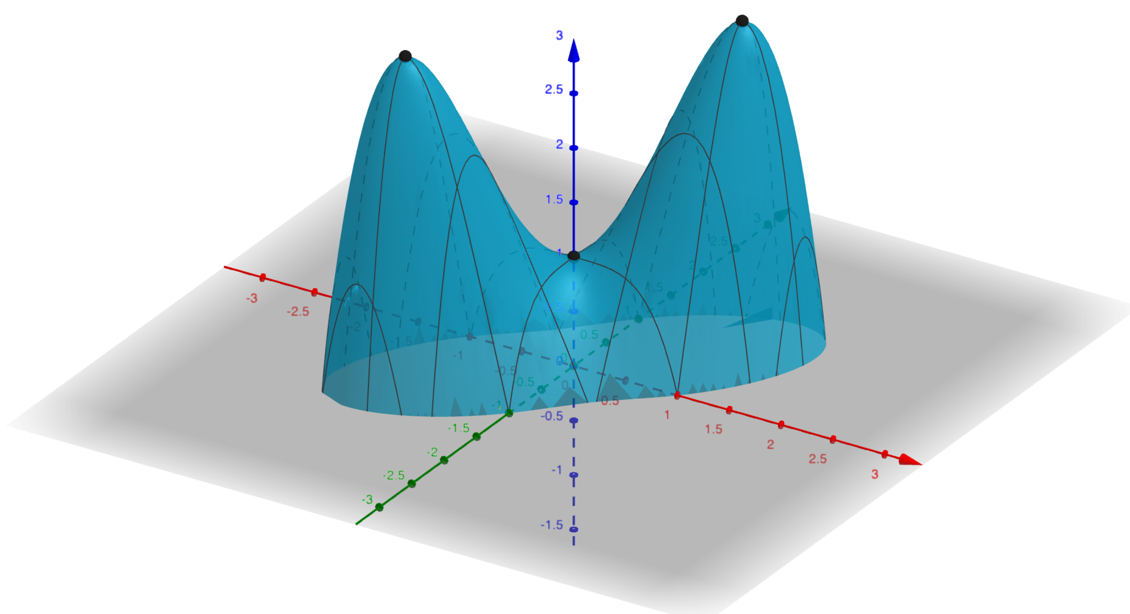
$\det(Hf)(-1,-1) = 144-16 > 0$,

so they are maxima or minima.

To distinguish between these we may consider f_{xx} or f_{yy}.

Since $f_{xx}(1,1) = -12 < 0$ we find that f is "curving down" at $(1,1)$ so $(1,1)$ is a <u>local max</u>.

And since $f_{xx}(-1,-1) = -12 < 0$ we see that $(-1,-1)$ is also a <u>local max</u>.



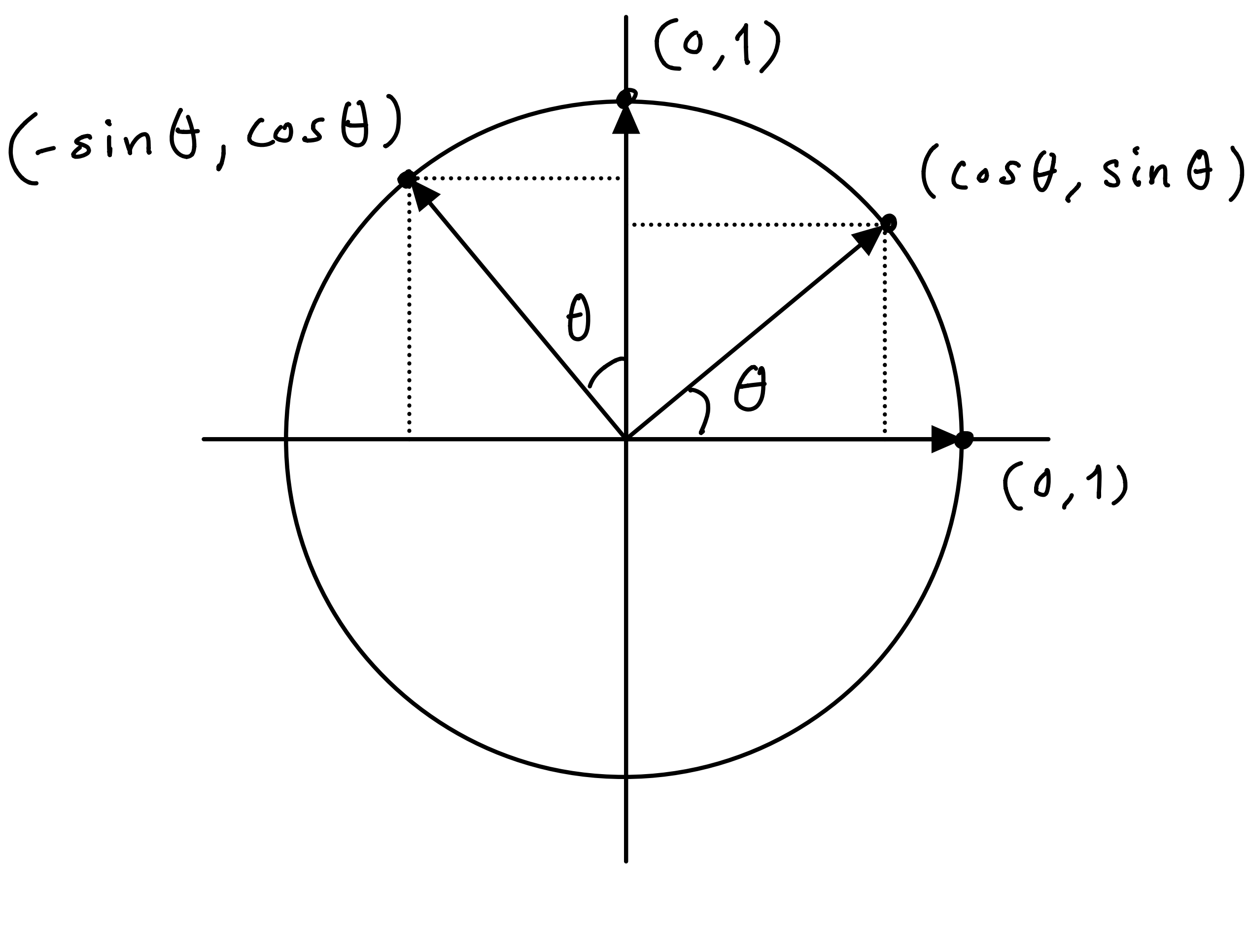
but not

close
exact

Discussion: This example is related to the example from class by "rotation". To rotate the coordinate system by angle θ, counterclockwise, we replace (x, y) by

$$(x\cos\theta - y\sin\theta,\ x\sin\theta + y\cos\theta).$$

Why?



Let $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be the function that rotates the plane by angle θ, counterclockwise. The above picture shows that

$$R_\theta(1,0) = (\cos\theta, \sin\theta),$$
$$R_\theta(0,1) = (-\sin\theta, \cos\theta).$$

But rotation is a "linear function", meaning that it preserves vector addition & scalar multiplication.

It follows that for any point (x,y) we have

$$R_\theta(x,y) = R_\theta\left[x(1,0) + y(0,1)\right]$$
$$= x R_\theta(1,0) + y R_\theta(0,1)$$
$$= x(\cos\theta, \sin\theta) + y(-\sin\theta, \cos\theta)$$
$$= (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$$

Example: To rotate by $45°$, send

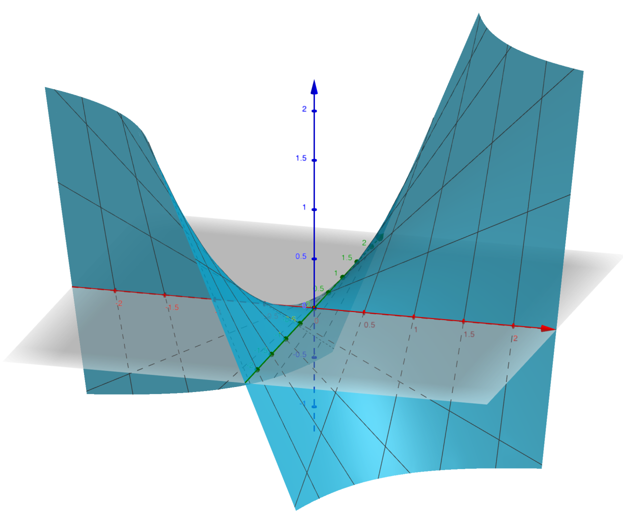
$$(x,y) \to \left(x\frac{1}{\sqrt{2}} - y\frac{1}{\sqrt{2}},\ x\frac{1}{\sqrt{2}} + y\frac{1}{\sqrt{2}} \right)$$

$$= \frac{1}{\sqrt{2}}(x-y,\ x+y)$$

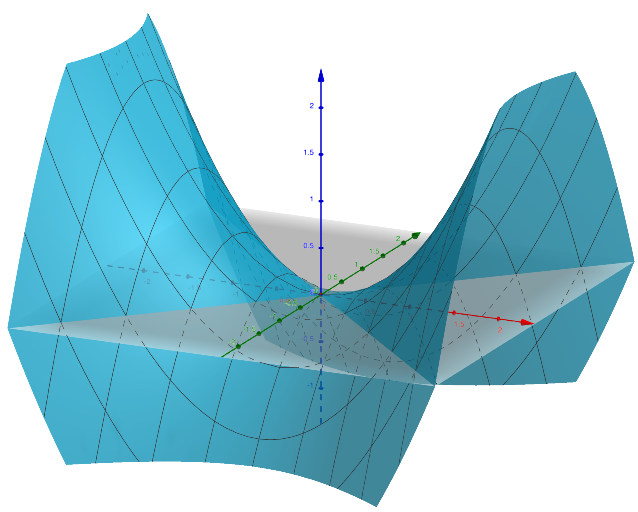
For example, if $g(x,y) = xy$ then the rotated field is

$$g\left(\frac{x-y}{\sqrt{2}}, \frac{x+y}{\sqrt{2}}\right) = \frac{x^2}{2} - \frac{y^2}{2}.$$

Pictures:



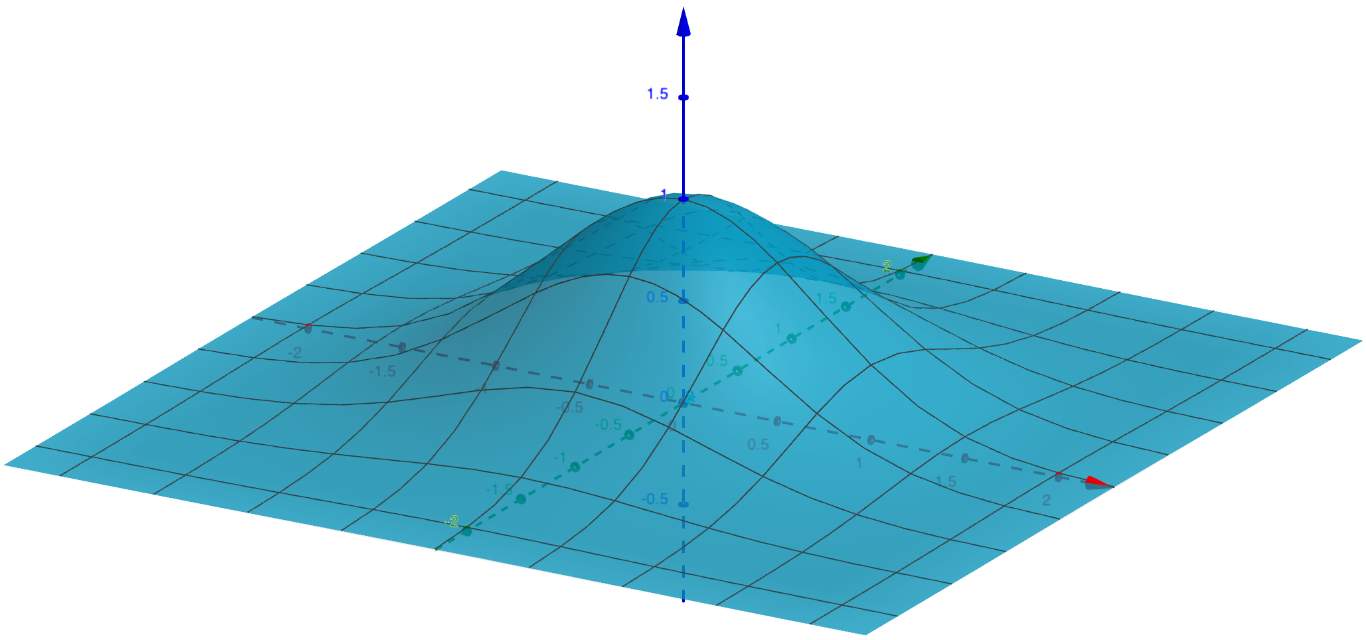
$z = xy$



$z = \frac{1}{2}x^2 - \frac{1}{2}y^2$

4. Directional Derivatives.

Let $f(x,y) = e^{-x^2-y^2}$ represent the temperature at the point (x,y) in the plane. We can visualize this by thinking of temperature as "height":



[Remark: This is called a "normal distribution", or a "Gaussian distribution". It is very important in physics and in statistics.]

If we move in the plane then our temperature will change. If our position is $\vec{r}(t) = \langle x(t), y(t) \rangle$ then our temperature at time t is

$$f(\vec{r}(t)) = f(x(t), y(t))$$

$$= e^{-x(t)^2 - y(t)^2}$$

a) Suppose we start at $\vec{r}(0) = \langle 1, 1 \rangle$ and travel with constant velocity

$$\vec{r}\,'(0) = \langle \cos\theta, \sin\theta \rangle,$$

so that $\vec{r}(t) = \langle 1 + t\cos\theta, 1 + t\sin\theta \rangle$.

[Note that our constant speed is

$$\|\vec{r}\,'(0)\| = \cos^2\theta + \sin^2\theta = 1. \text{]}$$

To compute our temperature at time t, Let's first compute

$$x(t)^2 + y(t)^2 = (1 + t\cos\theta)^2 + (1 + t\sin\theta)^2$$

$$= 1 + 2t\cos\theta + t^2\cos^2\theta$$
$$+ 1 + 2t\sin\theta + t^2\sin^2\theta$$

$$= 2 + 2t(\cos\theta + \sin\theta) + t^2$$

Thus we have

$$f(\vec{r}(t)) = e^{-x(t)^2 - y(t)^2}$$

$$= e^{-2 - 2t(\cos\theta + \sin\theta) + t^2}$$

and our rate of change of temperature is

$$f(\vec{r}(t))' = \frac{d}{dt} e^{-2 - 2t(\cos\theta + \sin\theta) + t^2}$$

$$= e^{-2 - 2t(\cos\theta + \sin\theta) + t^2} \cdot \left[2t - 2\cos\theta - 2\sin\theta\right]$$

At time $t = 0$ this becomes

$$f(\vec{r}(t))' \Big|_{t=0} = e^{-2}(-2\cos\theta - 2\sin\theta)$$

As we pass through (1,1) at constant speed 1, this is our rate of change of temperature.

b) For which angle is this rate of change maximized?

Let $g(\theta) = -2e^{-2}(\cos\theta + \sin\theta)$

$$\frac{dg}{d\theta} = -2e^{-2}(-\sin\theta + \cos\theta)$$

Set $dg/d\theta = 0$ to get

$$\sin\theta = \cos\theta$$

$$\longrightarrow \quad \theta = \pi/4 \text{ or } 5\pi/4$$

It turns out that $\theta = 5\pi/4$ is a max & $\theta = \pi/4$ is a min.

Alternate Method: we can use the multivariable chain rule.

If you travel with constant velocity $\vec{r}\,'(t) = \vec{v}$ through the point $\vec{p}$ (let's say $\vec{r}(0) = \vec{p}$), then your rate of change of temperature at time zero is

$$
\begin{aligned}
(f \circ \vec{r})'(0) &= f(\vec{r}(t))' \Big|_{t=0} \\
&= \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t) \\
&= \nabla f(\vec{p}) \cdot \vec{r}\,'(0) \\
&= \nabla f(\vec{p}) \cdot \vec{v}
\end{aligned}
$$

Jargon: This is called the "directional derivative of the field f at the point $\vec{p}$, in the direction of $\vec{v}$."

When is it maximized ?

Let α be the angle between your velocity $\vec{v}$ at $\vec{p}$ and the gradient vector $\nabla f(\vec{p})$ at $\vec{p}$, so that

$$\nabla f(\vec{p}) \cdot \vec{v} = \|\nabla f(\vec{p})\| \, \|\vec{v}\| \cos\alpha$$

If $\|\nabla f(\vec{p})\|$ & $\|\vec{v}\|$ are constant then this maximized when $\cos\alpha = +1$, i.e. when $\alpha = 0$.

In words: Your rate of change of temperature is maximized when your velocity points in the same direction as the gradient vector!

In our example:

$$\nabla f(x,y) = \left\langle -2x e^{-x^2-y^2}, -2y e^{-x^2-y^2} \right\rangle$$

$$\nabla f(1,1) = \left\langle -2e^{-2}, -2e^{-2} \right\rangle$$

and $\vec{v} = \vec{r}\,'(0) = \langle \cos\theta, \sin\theta \rangle$,

so the derivative of f at $\vec{p} = (1,1)$ in the direction of $\vec{v} = \langle \cos\theta, \sin\theta \rangle$ is

$$\nabla f(\vec{p}) \cdot \vec{v} = \langle -2e^{-2}, -2e^{-2} \rangle \cdot \langle \cos\theta, \sin\theta \rangle$$
$$= -2e^{-2}(\cos\theta + \sin\theta)$$

as before.

If we let α be the angle between $\nabla f(1,1)$ & $\vec{v}$ then we can also write

$$\nabla f(\vec{p}) \cdot \vec{v} = \|\nabla f(\vec{p})\| \, \|\vec{v}\| \cos\alpha$$
$$= \sqrt{4e^{-4} + 4e^{-4}} \cdot 1 \cos\alpha,$$

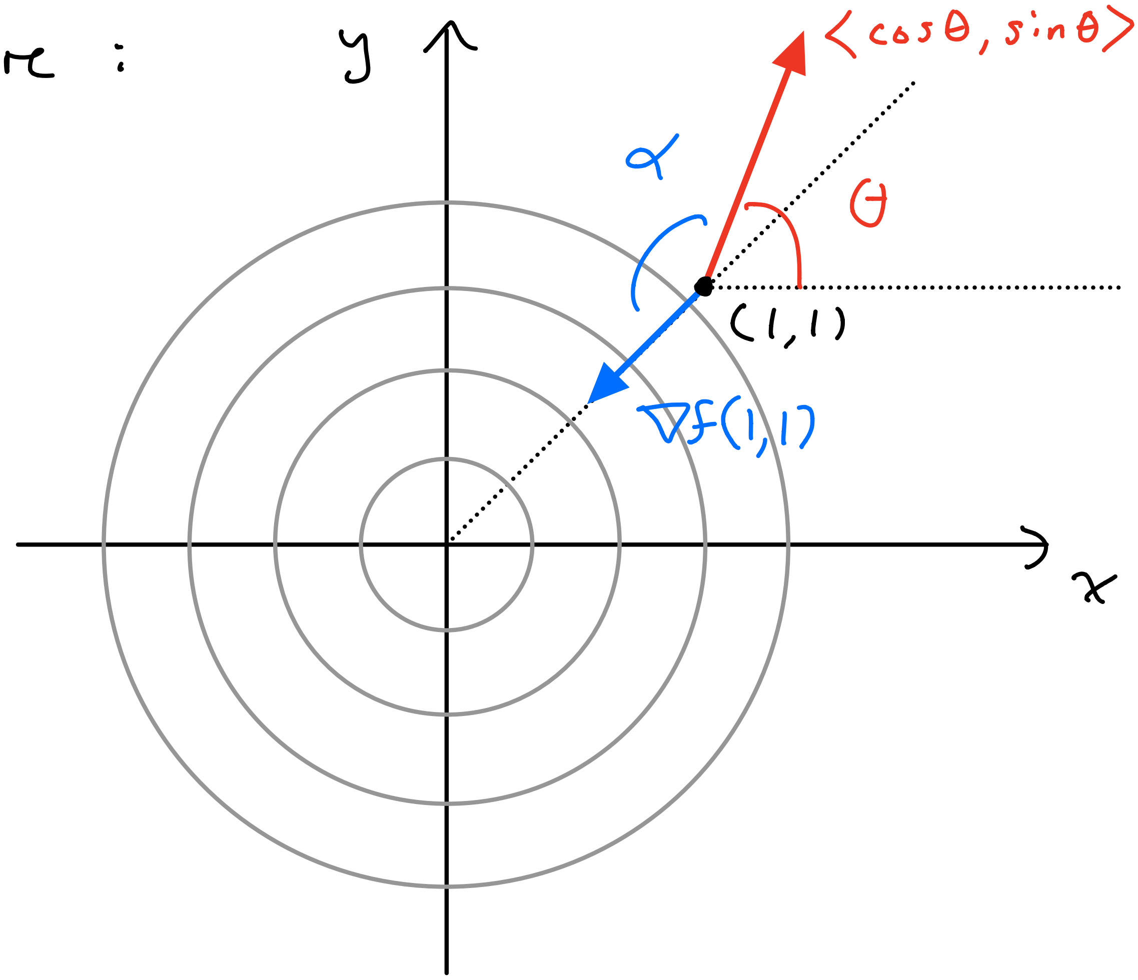
which is maximized when $\alpha = 0$.

The relationship between the two angles is

$$\theta + \alpha = \frac{5\pi}{4}$$

so that $\alpha = 0$ corresponds to $\theta = \frac{5\pi}{4}$.

Picture :



The gradient points toward the origin, so our velocity should point toward the origin.

5. Gradient Flow. The height of a hill above the point (x,y) is

$$f(x,y) = 100 - 2x^2 - y^2.$$

We want to find a path

$$\vec{r}(t) = \langle x(t), y(t) \rangle$$

with initial postition $\vec{r}(0) = \langle 9, 3 \rangle$ that always travels directly uphill. That is, we want the velocity $\vec{r}\,'(t)$ to point in the same direction as the gradient vector $\nabla f(\vec{r}(t))$ for all times t.

It is not clear how to find such a path. I will just give you two possible solutions.

a) Let $\vec{r}(t) = \langle 9e^{-4t}, 3e^{-2t} \rangle$ so that $\vec{r}(0) = \langle 9, 3 \rangle$. The velocity at time t is

$$\vec{r}\,'(t) = \langle 9(-4)e^{-4t}, 3(-2)e^{-2t} \rangle$$

$$= \langle -36\, e^{-4t}, -6\, e^{-2t} \rangle.$$

The gradient at the point (x,y) is

$$\nabla f = \langle -4x, -2y \rangle$$

so the gradient at $\vec{r}(t) = \langle 9\, e^{-4t}, 3\, e^{-2t} \rangle$ is

$$\nabla f(\vec{r}(t)) = \langle -4 \cdot 9 e^{-4t}, -2 \cdot 3\, e^{-2t} \rangle$$
$$= \langle -36\, e^{-4t}, -6\, e^{-2t} \rangle,$$

which is exactly equal to $\vec{r}\,'(t)$ ✓

b) Let $\vec{r}(t) = \langle (3-t)^2, 3-t \rangle$

so that $\vec{r}(0) = \langle 9, 3 \rangle$. The velocity at time t is

$$\vec{r}\,'(t) = \langle -2(3-t), -1 \rangle$$

The gradient at the point $\vec{r}(t)$ is

$$\nabla f(\vec{r}(t)) = \langle -4x, -2y \rangle$$
$$= \langle -4(3-t)^2, -2(3-t) \rangle$$

Does this point in the same direction as the velocity? Yes, because

$$\vec{r}\,'(t) = \langle -2(3-t), -1 \rangle$$

$$2(3-t)\,\vec{r}\,'(t) = \langle -4(3-t)^2, -2(3-t) \rangle$$

$$\underbrace{2(3-t)}_{\text{this is just some scalar}}\,\vec{r}\,'(t) = \nabla f(\vec{r}(t)).$$

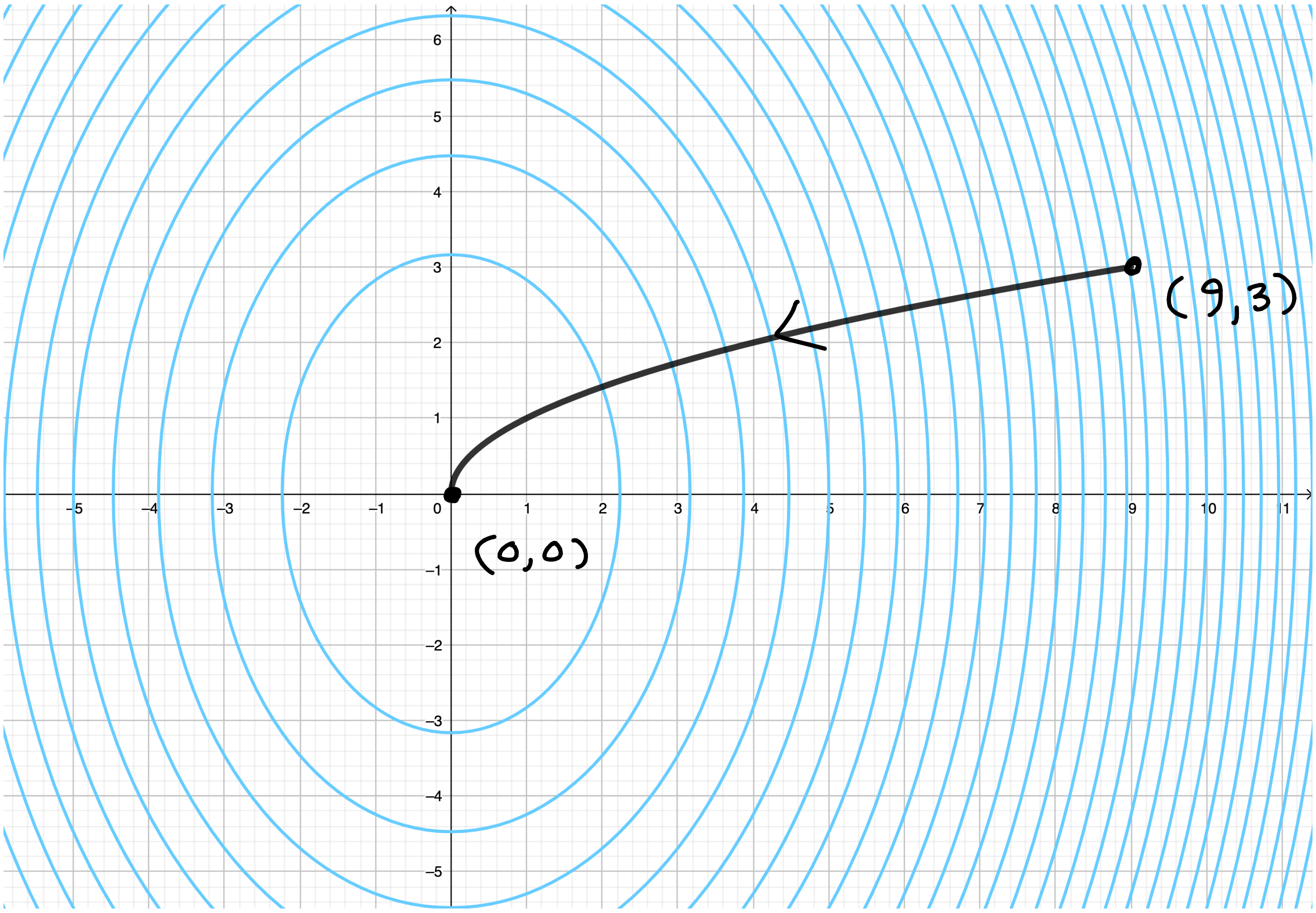
Remark: We say that vectors $\vec{u}$ & $\vec{v}$ are parallel if there exists a positive scalar k such that

$$k\vec{u} = \vec{v}.$$

If $k < 0$ then we say that $\vec{u}$ & $\vec{v}$ are "anti-parallel" (pointing in opposite directions). ///

If $0 \leq t < 3$ then $2(3-t) > 0$, so $\vec{r}\,'(t)$ & $\nabla f(\vec{r}(t))$ are parallel.

Picture :



The curve looks the same in both cases; it is the parabola $y^2 = x$. Only the speed changes.