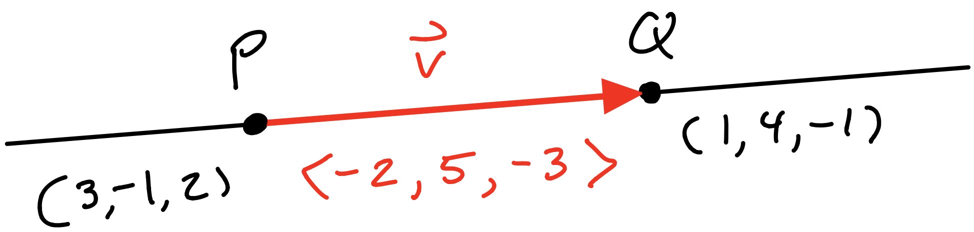
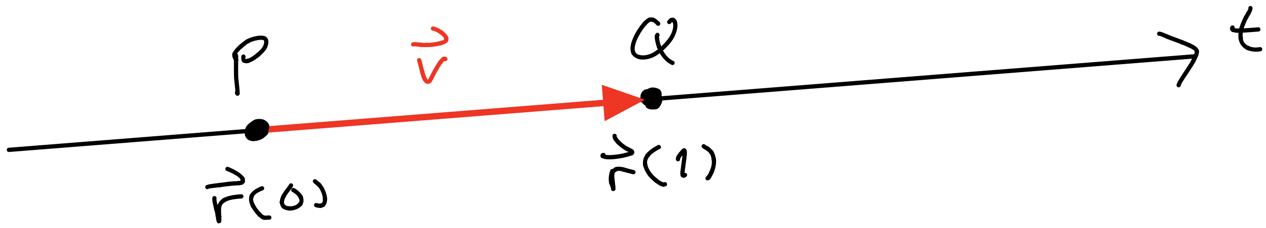
1. Consider the line passing through points $P = (3,-1,2)$ & $Q = (1,4,-1)$.

(a) To parametrize the line, we consider the vector $\vec{v} = \overrightarrow{PQ} = \langle a, b, c \rangle$:



If we take $(x_0, y_0, z_0) = P = (3,-1,2)$ then the line can be parametrized as

$$\vec{r}(t) = (3-2t, -1+5t, 2-3t)$$



(b) The parametrized line can be expressed as

$$\begin{cases} x = 3-2t \\ y = -1+5t \\ z = 2-3t. \end{cases}$$

To obtain the symmetric equations we solve for t:

$$t = \frac{x-3}{-2} = \frac{y+1}{5} = \frac{z-2}{-3}$$

Our line is the intersection of any two of the following three planes:

- $(x-3)/-2 = (y+1)/5$
- $(x-3)/-2 = (z-2)/-3$
- $(y+1)/5 = (z-2)/-3$.

2. Consider the following two planes

$$\begin{matrix} ① \\ ② \end{matrix} \begin{cases} x - y + 0 = 1, \\ x + y + 2z = 1. \end{cases}$$

To find the line of intersection we will let $t = z$ be the parameter. Then we solve for x & y.

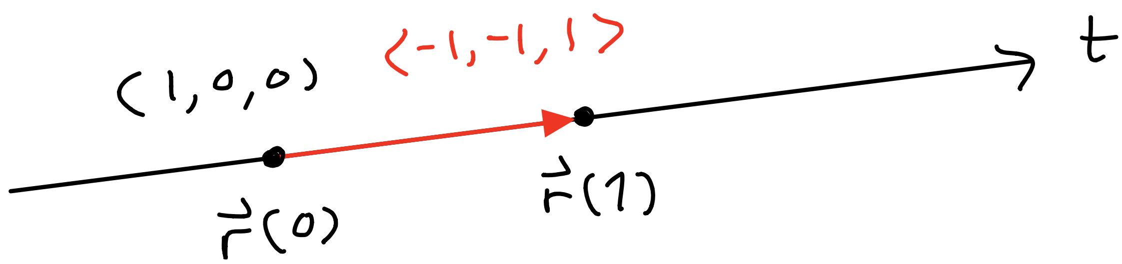
$$\begin{aligned}
①: \quad x - y &= 1 \\
②: \quad x + y &= 1 - 2t \\
① + ②: \quad 2x &= 2 - 2t \\
② - ①: \quad 2y &= -2t
\end{aligned}$$

We conclude that

$$\begin{cases} x = 1 - t \\ y = -t \\ z = t \end{cases}$$

Thus the line has parametrization

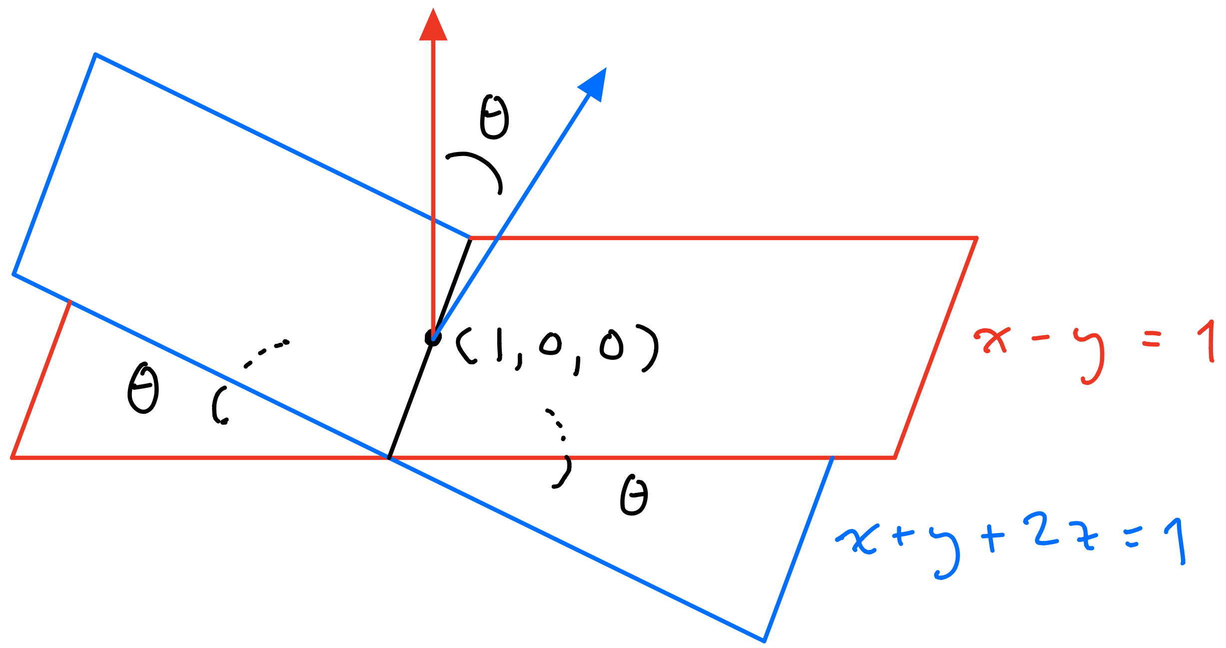
$$\vec{r}(t) = (1-t, -t, t)$$



Remark: Observe that $\langle -1,-1,1 \rangle$ is $\perp$ to both of the normal vectors $\langle 1,-1,0 \rangle$ & $\langle 1,1,2 \rangle$ as it should be.

(b) Find the angle between the planes.

Let $\vec{m} = \langle 1,-1,0\rangle$ & $\vec{n} = \langle 1,1,2\rangle$ be the normal vectors:



The angle between the planes is the same as the angle between $\vec{m}$ & $\vec{n}$, so that

$$\|\vec{m}\|\,\|\vec{n}\|\cos\theta = \vec{m}\cdot\vec{n}$$

$$\sqrt{1^2+(-1)^2+0^2}\,\sqrt{1^2+1^2+2^2}\cos\theta = 1\cdot 1+(-1)\cdot 1+0\cdot 2$$

$$\sqrt{2}\,\sqrt{6}\cos\theta = 0$$

$$\cos\theta = 0$$

It follows that $\theta = 90°$ so the planes are perpendicular. [The picture above is not very accurate but it is better for illustrating the general concept.]

3. An interesting curve

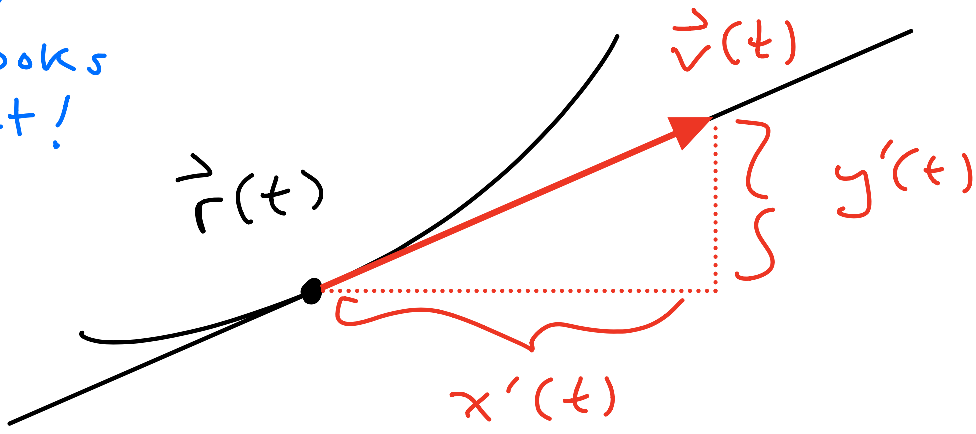
$$\vec{r}(t) = \langle x(t), y(t) \rangle = \langle t^2 - 1, t^3 - t \rangle$$

$$\vec{v}(t) = \langle x'(t), y'(t) \rangle = \langle 2t, 3t^2 - 1 \rangle$$

The slope of the tangent line at the point $\vec{r}(t)$ is

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y'(t)}{x'(t)} = \frac{3t^2-1}{2t}$$

this looks correct!



(a) The slope is vertical when

$$x'(t) = 0$$
$$2t = 0$$
$$t = 0$$

so that $\vec{r}(0) = (-1, 0)$

$$\vec{v}(0) = (0, -1)$$

The slope is horizontal when

$$y'(t) = 0$$
$$3t^2 - 1 = 0$$
$$t = \pm 1/\sqrt{3} \approx \pm 0.6$$

so that $\vec{r}(\pm 1/\sqrt{3}) \approx (-0.7, \pm 0.4)$

$$\vec{v}(\pm 1/\sqrt{3}) \approx (\pm 1.2, 0)$$

(b) The slope is $+1$ when

$$(3t^2 - 1)/2t = +1$$
$$3t^2 - 1 = 2t$$
$$3t^2 - 2t - 1 = 0$$
$$(3t + 1)(t - 1) = 0$$
$$t = 1 \text{ or } -1/3$$

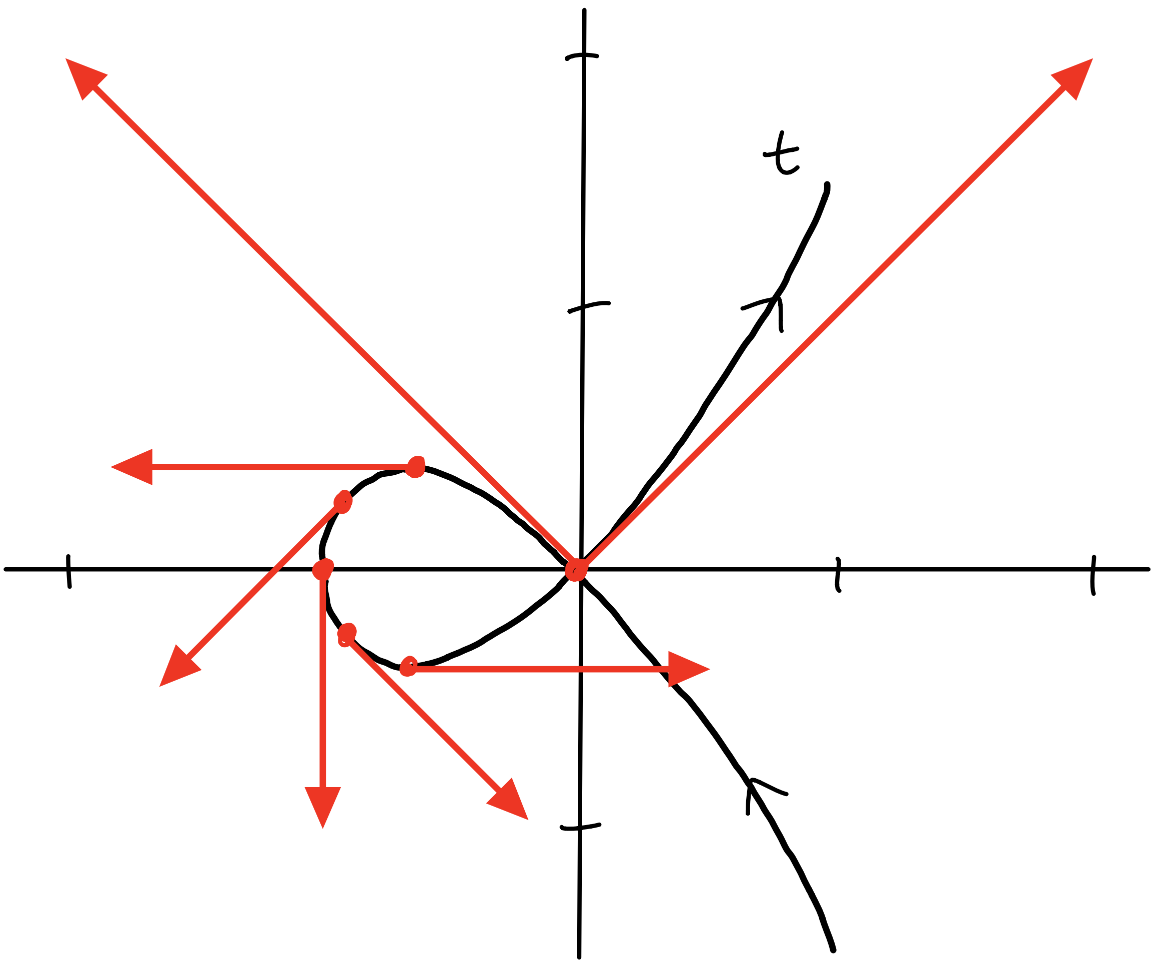
so that

$$\vec{r}(1) = (0,0) \qquad \text{or} \qquad \vec{r}(-1/3) \approx (-0.9, 0.3)$$
$$\vec{v}(1) = (2,2) \qquad \qquad \vec{v}(-1/3) \approx (-0.7, -0.7)$$

The slope is -1 when $t = -1$ or $t = +1/3$ [similar calculation] so that

$$\vec{r}(-1) = (0,0) \qquad \text{or} \qquad \vec{r}(1/3) \approx (-0.9, -0.3)$$
$$\vec{v}(-1) = (-2,2) \qquad \qquad \vec{v}(1/3) \approx (0.7, -0.7)$$

(c) Sketch



4. Projectile Motion

A projectile is launched from $(0,0)$ with initial speed 100 ft/sec and angle θ above the horizontal, so

$$\vec{r}(0) = \langle 0,0 \rangle,$$

$$\vec{v}(0) = \langle 100\cos\theta , 100\sin\theta \rangle.$$

The acceleration due to gravity is

$$\vec{a}(t) = \langle 0, -32 \rangle \text{ for all } t.$$

(a) By integrating twice we have

$$\vec{v}(t) = \langle 100\cos\theta , -32t + 100\sin\theta \rangle$$
$$\vec{r}(t) = \langle 100t\cos\theta , -16t^2 + 100t\sin\theta \rangle$$

[see the lecture notes for details.]

(b) The projectile hits the ground when

$$-16t^2 + 100t\sin\theta = 0$$

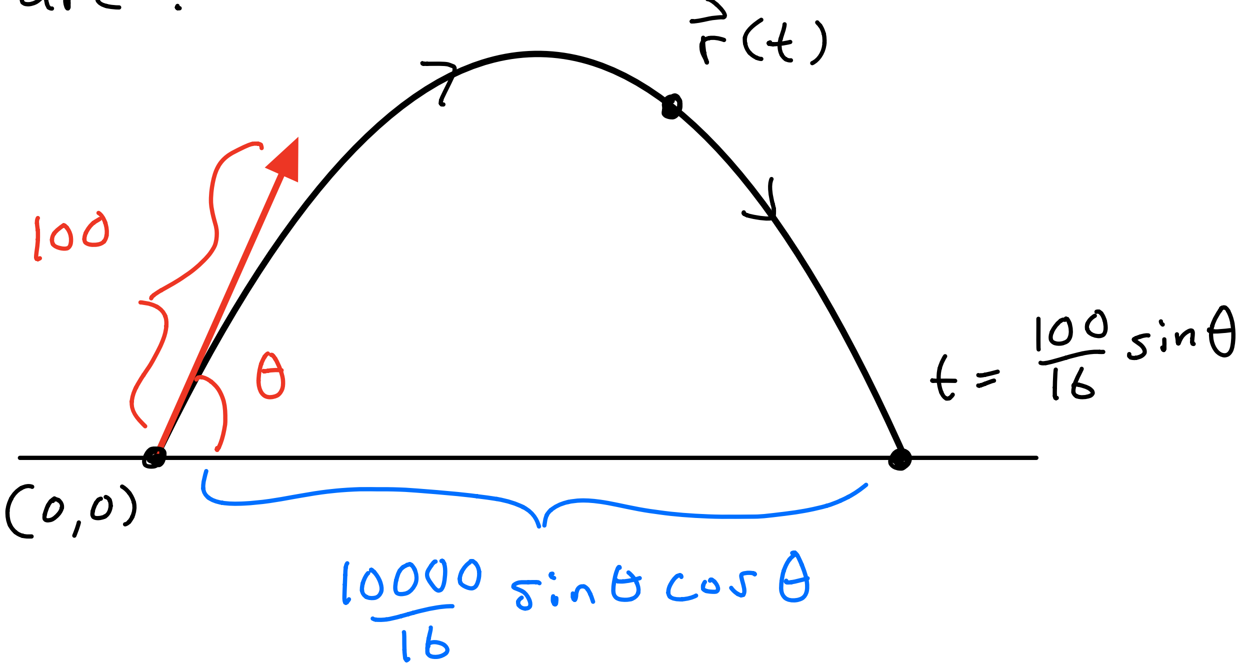
$$t(-16t + 100\sin\theta) = 0,$$

so $t = 0$ or $t = 100 \sin\theta / 16$.

The horizontal distance traveled is

$$100\, t \cos\theta = \frac{10000}{16} \sin\theta \cos\theta$$

Picture:



We want to maximize the distance as a function of θ, so we set the derivative equal to zero:

$$\frac{d}{d\theta}\left(\frac{10000}{16}\sin\theta\cos\theta\right) = 0$$

$$\frac{10000}{16}\left[\cos\theta\cos\theta - \sin\theta\sin\theta\right] = 0$$

$$\cos^2\theta - \sin^2\theta = 0$$

$$\cos^2\theta = \sin^2\theta$$

assume $\theta \neq 90^\circ$

$\theta \neq 0^\circ$

$$\tan^2\theta = 1$$

$$\tan\theta = \pm 1$$

So $\theta = 45^\circ$ or ~~135°~~ assume $\theta < 90^\circ$

The maximum possible distance is

$$\frac{10000}{16}\cos(45^\circ)\sin(45^\circ) = 312.5 \text{ feet}$$

[Remark: Actually the distance is always maximized at $\theta = 45^\circ$ for any initial speed s. In the general case, the distance traveled is $s^2\cos\theta\sin\theta/16$, so the maximum possible distance is $s^2/32$, when $\theta = 45^\circ$.]

5. Vector Identities.

(a) Given a vector $\vec{r}$ living in 137-dimensional space, let us define a new vector

$$\vec{u} = \frac{1}{\|\vec{r}\|}\,\vec{r}$$

scalar vector

To compute the length of $\vec{u}$ we use the dot product formula:

$$\|\vec{u}\|^2 = \vec{u}\cdot\vec{u}$$

$$= \left(\frac{1}{\|\vec{r}\|}\vec{r}\right)\cdot\left(\frac{1}{\|\vec{r}\|}\vec{r}\right)$$

$$= \frac{1}{\|\vec{r}\|}\cdot\frac{1}{\|\vec{r}\|}\left(\vec{r}\cdot\vec{r}\right)$$

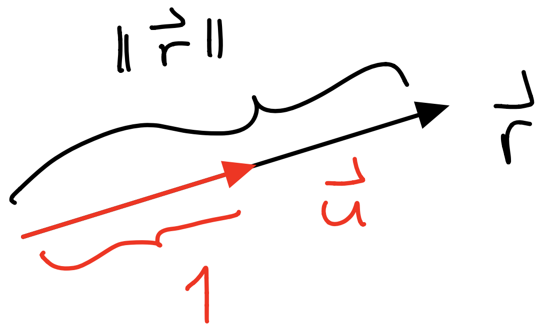
$$= \frac{1}{\|\vec{r}\|}\cdot\frac{1}{\|\vec{r}\|}\,\|\vec{r}\|^2 = 1$$

We conclude that

$$\|\vec{u}\|^2 = 1$$

$$\|\vec{u}\| = 1$$

Jargon: We call $\vec{u} = \vec{r}/\|\vec{r}\|$ a "unit vector" in the direction of $\vec{r}$.



(b) For any $\vec{v} = \langle v_1, v_2, v_3 \rangle$ we have that

$$\vec{v} \times \vec{v} = \langle v_1, v_2, v_3 \rangle \times \langle v_1, v_2, v_3 \rangle$$

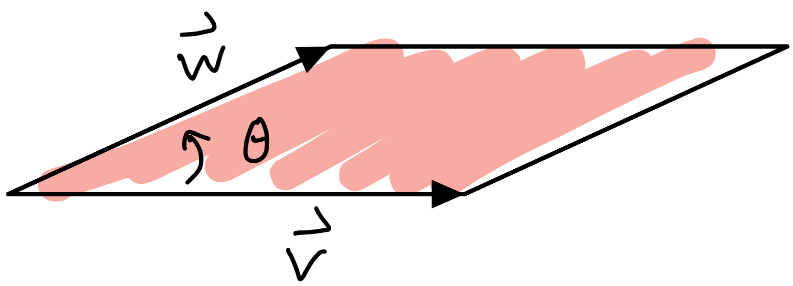
$$= \langle v_2 v_3 - v_3 v_2, v_3 v_1 - v_1 v_3, v_1 v_2 - v_2 v_1 \rangle$$

$$= \langle 0, 0, 0 \rangle \quad \checkmark$$

Remark: There is a more satisfying geometric explanation for this. Given vectors $\vec{v}, \vec{w}$ in $\mathbb{R}^3$, I claim that

$$\|\vec{v} \times \vec{w}\| = \|\vec{v}\| \|\vec{w}\| \sin\theta$$

= area of the parallelogram spanned by $\vec{v}, \vec{w}$.



[The proof is tricky so we won't discuss it.]

Since the angle between $\vec{v}$ and itself is 0 we get

$$\|\vec{v} \times \vec{v}\| = \|\vec{v}\| \|\vec{v}\| \sin 0 = 0$$

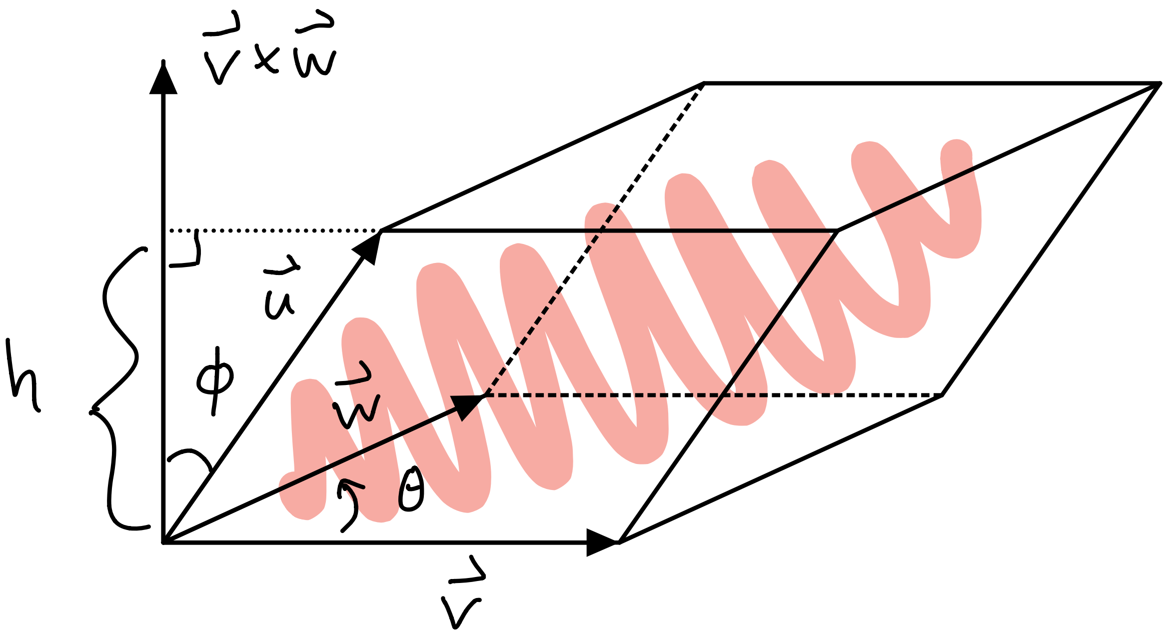
which implies that

$$\vec{v} \times \vec{v} = \langle 0,0,0 \rangle$$

[The only vector of length zero is the zero vector.]

This is also related to the "scalar triple product"

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = \pm \text{Volume of parallelepiped}$$



The volume always equals

$$\begin{aligned}
\text{vol} &= \text{height} \cdot (\text{area of base}) \\
&= h \, \|\vec{v} \times \vec{w}\| \\
&= (\|\vec{u}\| \cos \phi) \, \|\vec{v} \times \vec{w}\| \\
&= \|\vec{u}\| \, \|\vec{v} \times \vec{w}\| \cos \phi \\
&= \vec{u} \cdot (\vec{v} \times \vec{w})
\end{aligned}$$

If $\vec{u}$ is pointing below the plane generated by $\vec{v}, \vec{w}$ (i.e. if $\vec{v}, \vec{w}, \vec{u}$ is a "left handed system") then actually $h = -\|\vec{u}\| \cos \phi$ and

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = -\text{Vol}$$

(c) Suppose that a particle moves on the surface of a sphere of radius c:

$$\|\vec{r}(t)\| = c = \text{constant}$$

Then I claim that the velocity is always tangent to the sphere.

Proof: We have

$$\|\vec{r}(t)\|^2 = c^2$$

$$\vec{r}(t) \cdot \vec{r}(t) = c^2$$

Take the time derivative of both sides:

$$(\vec{r}(t) \cdot \vec{r}(t))' = (c^2)'$$

$$\vec{r}\,'(t) \cdot \vec{r}(t) + \vec{r}(t) \cdot \vec{r}\,'(t) = 0$$

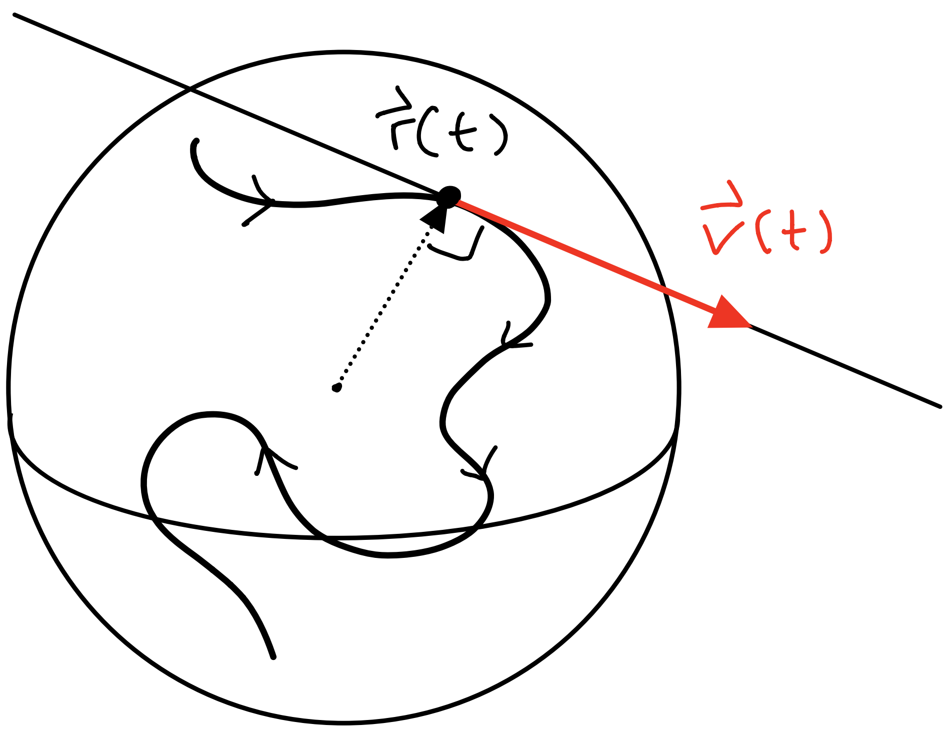
$$2\,\vec{r}(t) \cdot \vec{r}\,'(t) = 0$$

$$\vec{r}(t) \cdot \vec{r}\,'(t) = 0$$

for all times t

In other words, the position vector $\vec{r}(t)$ (which is a radius of the sphere) and the velocity vector $\vec{r}\,'(t) = \vec{v}(t)$

are always perpendicular :



We used this in our computations yesterday when we said that

$\|\vec{u}(t)\| = 1$ for all t implies

that $\vec{u}(t) \cdot \vec{u}\,'(t) = 0$ for all t.