PRACTICE PROBLEMS FOR EXAM 1

1. Let $\mathbf{v} = 2\mathbf{i} + 2\mathbf{k}$ and $\mathbf{w} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$. (a) Find $|\mathbf{v}|$. (b) Find the unit vector with the same direction as $\mathbf{v}$. (c) Find $\mathbf{v} \cdot \mathbf{w}$. (d) State a formula for the angle between two vectors. (e) Find the angle between $\mathbf{v}$ and $\mathbf{w}$. (f) Find $\mathbf{v} + \mathbf{w}$. (g) Find the component of $\mathbf{w}$ in the direction of $\mathbf{v}$. (h) Write $\mathbf{w}$ as the sum of two vectors, one of them parallel to $\mathbf{v}$, the other perpendicular to $\mathbf{v}$. (i) Find $\mathbf{v} \times \mathbf{w}$.

2. True or false ? (a) $\mathbf{u} \times \mathbf{v} = 0$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are perpendicular. (b) $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{u} \times (\mathbf{v} \cdot \mathbf{w})$ for all vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ in space. (c) If $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{w}$ and $\mathbf{u} \neq 0$ then $\mathbf{v} = \mathbf{w}$.

3. (a) Check that the two lines with the parametric equations $\mathbf{r}_1(t) = \langle 0, 2, -3\rangle + t\langle 1, 0, -1\rangle$ and $\mathbf{r}_2(s) = \langle -2, 3, -6\rangle + s\langle 3, -1, 2\rangle$ intersect at the point $P = (1, 2, -4)$. (b) What is the acute angle between the two lines? You can leave your answer in terms of arccos.

4. (a) Find a nonzero vector perpendicular to the vectors $\mathbf{u} = \mathbf{i} - \mathbf{j} + \mathbf{k}$ and $\mathbf{v} = 2\mathbf{j} - 3\mathbf{k}$. (b) Find the area of the parallelogram spanned by $\mathbf{u}$ and $\mathbf{v}$.

5. Find the equation of the plane passing through the point $(0, 1, -2)$ and containing the line $\mathbf{r}(t) = \langle 0, 2, -3\rangle + t\langle 1, 0, -1\rangle$.

6. Consider the parametric line $\mathbf{r}(t) = \langle -2, 3, -6\rangle + t\langle 3, -1, -2\rangle$ and the plane given by the equation $2x - y + 3z = 5$. (a) Show that the line intersects the plane at precisely one point and find its coordinates. (b) Find the acute angle that the line makes with a vector normal to the plane.

7. Determine if there exists a plane that contains all four points $P(3, 1, 2)$, $Q(6, -1, 6)$, $R(1, 4, 7)$, and $S(2, 1, 5)$.

8. Find symmetric equations for the line of intersection of the planes $x + y - z = 2$ and $3x - 4y + 5z = 6$.

9. The velocity of a particle moving in space is given by

$$\frac{d\mathbf{r}}{dt} = -\sin t \cdot \mathbf{i} + \cos t \cdot \mathbf{j} + 3\mathbf{k}.$$

(a) Find the position vector $\mathbf{r}(t)$ of the particle if $\mathbf{r}(0) = \mathbf{i} + 3\mathbf{k}$. (b) Find the unit tangent vector to the trajectory of the particle when $t = \pi$. (c) Find particle's acceleration when $t = \pi$. (d) Find the distance traveled by the particle along the curve $\mathbf{r}(t)$ from $t = 0$ to $t = \pi$.

10. The surface with equation $z+\cos(xy) = x^2+y^2$ can be described both as a level surface of a function f and as the graph of a function g. Give explicit formulas for f and g.

11. (a) Let $f(x, y, z) = \sin(xy + z)$. Find all first and second order partial derivatives of f.
(b) Evaluate

$$\frac{\partial^{100}}{\partial x^{95}\, \partial y^2\, \partial x^3}\left(ye^x x + \cos(x)\right).$$

12. Find the equation of the tangent plane to the graph $z = -x^2 + 4y^2 + 1$ at the point $(2, 1, 1)$.

13. Let $f(x, y)$ be a differentiable function such that $f(1, 1) = 3$, $f_x(1, 1) = 2$, and $f_y(1, 1) = -1$. From the available information, what is the best estimate you can give of $f(1.1,\ 0.9)$?

14. The radius of a right circular cone is measured at 120 in with a possible error of 1.8 in, while its height measured at 140 in with a possible error of 2.5 in. Estimate the maximal possible error if these measurements are used to compute the volume of this cone.

SOLUTIONS

1. (a) $2\sqrt{2}$; (b) $\left\langle \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right\rangle$; (c) 6; (e) $\frac{\pi}{6}$; (f) $\langle 4, 1, 3\rangle$;
(g) $\frac{3}{\sqrt{2}}$; (h) $\langle 1.5, 0, 1.5\rangle + \langle 0.5, 1, -0.5\rangle$; (i) $\langle -2, 2, 2\rangle$.

2. All three are false. **3.** (b) $\arccos\left(\frac{1}{2\sqrt{7}}\right)$. **4.** (a) $\langle 1, 3, 2\rangle$; (b) $\sqrt{14}$.

5. $x + y + z + 1 = 0$. **6.** (a) $P(88, -27, -66)$; (b) $\arccos\left(\frac{1}{14}\right)$.

7. There is no such plane. **8.** $x - 2 = y/(-8) = z/(-7)$.

9. (a) $\langle \cos t, \sin t, 3 + 3t\rangle$; (b) $\mathbf{T} = (-\mathbf{j} + 3\mathbf{k})/\sqrt{10}$; (c) $\mathbf{r}'' = \mathbf{i}$; (d) $\sqrt{10}\,\pi$.

10. $f(x, y, z) = x^2 + y^2 - \cos(xy) - z$, $\quad g(x, y) = x^2 + y^2 - \cos(xy)$.

11. (a) $f_x = y\cos(xy + z)$, $f_y = x\cos(xy + z)$, $f_z = \cos(xy + z)$; (b) 0.

12. $z = 1 - 4x + 8y$. **13.** $f(1.1, 0.9) \approx f(1, 1) + 2 \cdot 0.1 + (-1) \cdot (-0.1) = 3.3$.

14. $V(r, h) = \frac{1}{3}\pi r^2 h$, $\quad dV = V_r(120, 140)\, dr + V_h(120, 140)\, dh = 32160\,\pi$.