PRACTICE PROBLEMS FOR EXAM 2

1. Let $f(x,y)$ have continuous second partial derivatives, and let $x = st$ and $y = e^{st}$.
(a) Find $\partial x/\partial t$ and $\partial y/\partial t$.
(b) Find $\partial f/\partial t$ in terms of $\partial f/\partial x$, $\partial f/\partial y$, s and t.
(c) Find $\partial^2 f/\partial t^2$ in terms of $\partial^2 f/\partial x^2$, $\partial^2 f/\partial x\partial y$, $\partial^2 f/\partial y^2$, $\partial f/\partial x$, $\partial f/\partial y$, s and t.

2. Consider the function $f(x,y) = 3x^2 - xy + y^3$.
(a) Find the rate of change of f at $(1,2)$ in the direction of $\mathbf{v} = 3\mathbf{i} + 4\mathbf{j}$.
(b) In what direction (unit vector) does f decrease at $(1,2)$ at the maximum rate ? What is this maximum rate of change?
(c) In what directions is the rate of change of f at $(1,2)$ equal to zero? Your answer should be a pair of opposite unit vectors.

3. Suppose the gradient $\nabla f(2,4)$ of a function $f(x,y)$ has length equal to 5. Is there a direction $\mathbf{u}$ such that the directional derivative $D_{\mathbf{u}}f$ at the point $(2,4)$ is 7? Explain your answer.

4. Find the tangent plane to the ellipsoid $x^2+4y^2 = 169-9z^2$ at the point $P = (3,2,4)$.

5. Find the points on the surface (ellipsoid) $x^2 + 2y^2 + 4z^2 + xy + 3yz = 1$ where the tangent plane is parallel to the xz plane.

6. Find all the critical points of $f(x,y) = x^2 + y^2/2 + x^2y$ and apply the second derivative test to each of them.

7. Find the absolute maximum and minimum values of the function $f(x,y) = (x-1)^2 + (y-1)^2$ in the rectangular domain $D = \{(x,y) : 0 \le x \le 1, \ \ 0 \le y \le 2\}$. Justify your answer.

8. Find the maximum of $f(x,y) = xy$ restricted to the curve $(x+1)^2 + y^2 = 1$. Give both the coordinates of the point and the value of f.

9. Find the dimensions of a rectangular box of maximum volume such that the sum of the lengths of its 12 edges is a constant C.

10. Compute $\iint_D (3x+1)\,dxdy$ where D is the region in the first quadrant bounded by the parabolas $y = x^2$ and $y = (x-1)^2$ and the y–axis.

11. Change the order of integration in the following iterated integral:

$$\int_{-2}^{2} \int_{0}^{\sqrt{4-y^2}} f(x,y)\, dx\, dy.$$

12. (a) Find the area of the region enclosed by the cardioid given in polar coordinates by $r = 1 + \cos(\theta)$. (b) Use polar coordinates to evaluate

$$\int_{0}^{\sqrt{2}} \int_{y}^{\sqrt{4-y^2}} \frac{1}{1+x^2+y^2}\, dx\, dy.$$

SOLUTIONS

1. (a) $x_t = s,\ y_t = s\, e^{st}$; (b) $f_t = s\, f_x + s\, e^{st}\, f_y$;
(c) $f_{tt} = s^2\, e^{st}\, f_y + s^2\, f_{xx} + 2s^2\, e^{st}\, f_{xy} + s^2\, e^{2st}\, f_{yy}$.

2. (a) $\mathbf{u} = \dfrac{\mathbf{v}}{|\mathbf{v}|} = \dfrac{3}{5}\mathbf{i} + \dfrac{4}{5}\mathbf{j}$, $(D_{\mathbf{u}} f)(1,2) = (\nabla f)(1,2) \cdot \mathbf{u} = 56/5$.

(b) $-\dfrac{\nabla f}{|\nabla f|}(1,2) = \left(-\dfrac{4}{\sqrt{137}}, -\dfrac{11}{\sqrt{137}}\right)$. (c) $\left(\dfrac{11}{\sqrt{137}}, -\dfrac{4}{\sqrt{137}}\right), \left(-\dfrac{11}{\sqrt{137}}, \dfrac{4}{\sqrt{137}}\right)$.

3. (a) No, because $|(D_{\mathbf{u}} f)(2,4)| = |(\nabla f)(2,4)||\mathbf{u}| \cos\theta$ where θ is the angle between $\mathbf{u}$ and $(\nabla f)(2,4)$. If $|(D_{\mathbf{u}} f)(2,4)| = 7$ then $7 = 5\cos\theta$ which is impossible because $\cos\theta \le 1$.

4. $3x + 8y + 36z = 169$. **5.** The points where $\dfrac{\partial f}{\partial x} = \dfrac{\partial f}{\partial z} = 0$, i.e.

$$\left(-\frac{2}{\sqrt{19}}, \frac{4}{\sqrt{19}}, -\frac{3}{2\sqrt{19}}\right) \quad \text{and} \quad \left(\frac{2}{\sqrt{19}}, -\frac{4}{\sqrt{19}}, \frac{3}{2\sqrt{19}}\right).$$

6. $(0,0)$ is a local minimum, $(1,-1)$ and $(-1,-1)$ are saddles.

7. Minimum 0, maximum 2. **8.** Maximum $f(-3/2, -\sqrt{3}/2) = 3\sqrt{3}/4$.

9. $C/12$, $C/12$, $C/12$. **10.** $3/8$.

11. $\displaystyle\int_{0}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} f(x,y)\, dy\, dx$. **12.** (a) $3\pi/2$. (b) $\dfrac{\pi}{8} \ln 5$.

More practice problems

1. Let $f(x, y, z) = x + 2y + z$, and R the solid $x^2 + y^2 + z^2 \leq 4$, $\sqrt{3(x^2 + y^2)} \leq z$. Set up an integral to compute $\iiint_R f(x, y, z)\, dxdydz$ (a) using rectangular coordinates (x, y, z), (b) using cylindrical coordinates (r, θ, z), and (c) using spherical coordinates (ρ, θ, φ) (do not evaluate the integrals).

2. Find the volume of the region in space bounded by the paraboloid $x = 1 - y^2 - z^2$ and the plane $x = 0$.

3. The solid E in the first octant is obtained by removing the cylinder $x^2 + y^2 = 1$ from the sphere $x^2 + y^2 + z^2 = 4$. Set up a triple integral in cylindrical coordinates to compute the total mass of E if its density is given by $\rho(x, y, z) = z^2 + \sqrt{x^2 + y^2}$. Do not evaluate the integral.

4. (a) Find the volume of one of the wedges cut from the cylinder $x^2 + y^2 = a^2$ by the planes $z = 0$ and $z = mx$, $m > 0$. (b) Use spherical coordinates to evaluate

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} (x^2 + y^2 + z^2)^2 \, dz \, dy \, dx \, .$$

SOLUTIONS

1.

$$\text{(a)}\quad \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{\sqrt{3(x^2+y^2)}}^{\sqrt{4-x^2-y^2}} (x+2y+z)\,dzdydx,$$

$$\text{(b)}\quad \int_0^{2\pi}\int_0^1\int_{\sqrt{3}\,r}^{\sqrt{4-r^2}} r(r\cos\theta+2r\sin\theta+z)\,dzdrd\theta,$$

$$\text{(c)}\quad \int_0^{2\pi}\int_0^{\pi/6}\int_0^2 \rho^3(\sin\varphi\cos\theta+2\sin\varphi\sin\theta+\cos\varphi)\sin\varphi\,d\rho d\varphi d\theta.$$

2. $\pi/2$.

3.

$$\int_0^{\pi/2}\int_1^2\int_0^{\sqrt{4-r^2}} r(z^2+r)\,dzdrd\theta.$$

4. (a) $2ma^3/3$. (b) $\pi/14$.