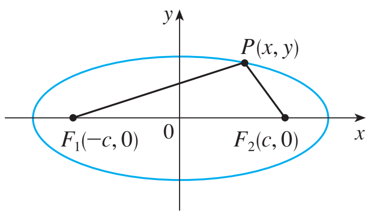

FIGURE 3

An ellipse has a simple equation if we place the foci on the x-axis at the points $(-c, 0)$ and $(c, 0)$ as in Figure 3 so that the origin is halfway between the foci. If the sum of the distances from a point on the ellipse to the foci is $2a$, then the points $(a, 0)$ and $(-a, 0)$ where the ellipse meets the x-axis are called the **vertices**. The y-intercepts are $\pm b$, where $b^2 = a^2 - c^2$. (See Figure 4.)

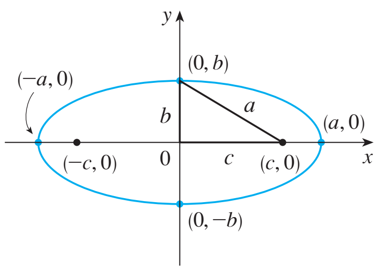

FIGURE 4
$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

1 The ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad a \geqslant b > 0$$

has foci $(\pm c, 0)$, where $c^2 = a^2 - b^2$, and vertices $(\pm a, 0)$.

If the foci of an ellipse are located on the y-axis at $(0, \pm c)$, then we can find its equation by interchanging x and y in **1**.

A **hyperbola** is the set of all points in a plane the difference of whose distances from two fixed points F_1 and F_2 (the foci) is a constant. Notice that the definition of a hyperbola is similar to that of an ellipse; the only change is that the sum of distances has become a difference of distances. If the foci are on the x-axis at $(\pm c, 0)$ and the difference of distances is $\pm 2a$, then the equation of the hyperbola is $(x^2/a^2) - (y^2/b^2) = 1$, where $b^2 = c^2 - a^2$. The x-intercepts are $\pm a$ and the points $(a, 0)$ and $(-a, 0)$ are the **vertices** of the hyperbola. There is no y-intercept and the hyperbola consists of two parts, called its *branches.* (See Figure 5.)

When we draw a hyperbola it is useful to first draw its **asymptotes**, which are the dashed lines $y = (b/a)x$ and $y = -(b/a)x$ shown in Figure 5. Both branches of the hyperbola approach the asymptotes; that is, they come arbitrarily close to the asymptotes.

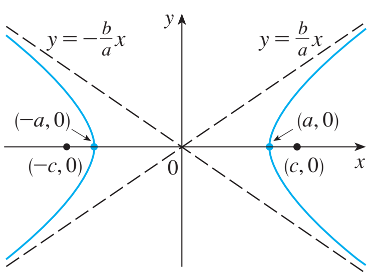

FIGURE 5
$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

2 The hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

has foci $(\pm c, 0)$, where $c^2 = a^2 + b^2$, vertices $(\pm a, 0)$, and asymptotes $y = \pm(b/a)x$.

If the foci of a hyperbola are on the y-axis, then by reversing the roles of x and y we get the graph shown in Figure 6.

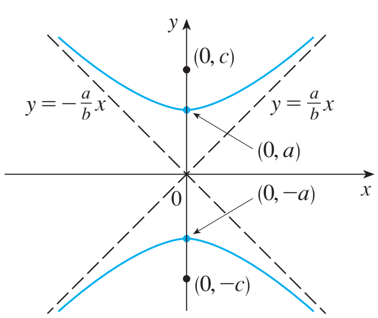

FIGURE 6
$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$

We have given the standard equations of the conic sections, but any of them can be shifted by replacing x by $x - h$ and y by $y - k$. For instance, an ellipse with center (h, k) has an equation of the form

$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$$

CONICS IN POLAR COORDINATES

In the following theorem we show how all three types of conic sections can be characterized in terms of a focus and directrix.