MTH 211: Calculus III — Fall 2019
Summary of Topics — Drew Armstrong

Mon Aug 19: 2D and 3D space (Sections 10.1 and 10.5)

- Cartesian plane, right handed Cartesian space
- Analytic geometry: Equations define shapes
- Equation of a line in 2D:

$$y = mx + b \text{ or } (y - y_0) = m(x - x_0) \text{ or } (y - y_0)/(x - x_0) = (y_1 - y_0)/(x_1 - x_0).$$

- Most general equation of a line in 2D: $ax + by = c$
- Pythagorean Theorem, distance between points in 2D
- Equation of a circle in 2D: $(x - a)^2 + (y - b)^2 = r^2$
- Equation of a sphere in 3D: $(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$ (why?)
- Pythagorean Theorem, distance between points in 3D
- Equation of a line in 3D? (does not exist)
- Equation of a plane in 3D? $ax + by + cz = d$ (why? we'll see next time)
- A line in 3D is described as the intersection of two planes,
- or it can be parametrized: $(x, y, z) = (x_0, y_0, z_0) + t(a, b, c)$.
- We like to write this as

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v},$$

  where $\mathbf{r} = \langle x, y, z\rangle$, $\mathbf{r}_0 = \langle x_0, y_0, z_0\rangle$ and $\mathbf{v} = \langle a, b, c\rangle$ are called vectors.

Wed Aug 21: 2D and 3D space, Vectors (Sections 10.1 and 10.2)

- Cartesian 2-space is called $\mathbb{R}^2$, Cartesian 3-space is called $\mathbb{R}^3$.
- What shape in $\mathbb{R}^2$ is defined by the equation $x = y$? In $\mathbb{R}^3$?
- Discuss Exercises 10.1: 6, 7, 10.
- What is a vector?
- In Stewart's book, round parentheses (a, b, c) denote a point in $\mathbb{R}^3$. Angle brackets $\langle a, b, c\rangle$ denote the arrow (directed line segment) with tail at the point $(0, 0, 0)$ and head at the point (a, b, c).
- The important thing about arrows is that they can be picked up and moved around. The arrow with tail at the point (a_1, b_1, c_1) and head the point (a_2, b_2, c_2) is always called $\langle a_2 - a_1, b_2 - b_1, c_2 - c_1\rangle$.

Mon Aug 26: Vectors and Dot Product (Sections 10.2 and 10.3)

- Vector addition, triangle and parallelogram
- Scalar multiplication and subtraction
- Exercise 10.2.4
- The set of points $\mathbf{x}_0 + t\mathbf{a}$, for all $t \in \mathbb{R}$, is a line.
- Standard basis vectors $\mathbf{i} = \langle 1, 0, 0\rangle$, $\mathbf{j} = \langle 0, 1, 0\rangle$, $\mathbf{k} = \langle 0, 0, 1\rangle$.
- Notation: $\langle a, b, c\rangle = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$.
- Example 10.2.5
- Vector $\mathbf{a} = \langle a, b, c\rangle$ has length $|\mathbf{a}| = \sqrt{a^2 + b^2 + c^2}$.
- Unit vector in the direction of $\mathbf{a}$
- Angle? Exercises 10.2: 19, 20
- Dot product defined by $\mathbf{a} \bullet \mathbf{b} = a_1b_1 + a_2b_2 + \cdots$.

- Angle between $\mathbf{a}$ and $\mathbf{b}$ satisfies: $\mathbf{a} \bullet \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$.

Wed Aug 28: HW1, Dot Product, Lines and Planes (Sections 10.3 and 10.5)

- HW1 Discussion (10.1.12, 10.1.20, 10.2.2, 10.2.14, 10.2.33, 10.3.11, 10.3.13)
- Length of $t\mathbf{a}$ is $|t||\mathbf{a}|$. Unit vectors in direction of $\mathbf{a}$ are $\pm\mathbf{a}/|\mathbf{a}|$
- The standard basis vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ satisfy $\mathbf{i} \bullet \mathbf{i} = \mathbf{j} \bullet \mathbf{j} = \mathbf{k} \bullet \mathbf{k} = 1$ and $\mathbf{i} \bullet \mathbf{j} = \mathbf{i} \bullet \mathbf{k} = \mathbf{j} \bullet \mathbf{k} = 0$. What does this mean geometrically?
- Recall: angle between $\mathbf{a}$ and $\mathbf{b}$ satisfies: $\mathbf{a} \bullet \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$.
- Special case: $\mathbf{a} \bullet \mathbf{b} = 0$ if and only if $\mathbf{a} \perp \mathbf{b}$. This lets us describe lines and planes.
- The line containing (x_0, y_0) and perpendicular to $\mathbf{a} = \langle a, b\rangle$ has equation
$$a(x - x_0) + b(y - y_0) = 0.$$
- The plane containing point (x_0, y_0, z_0) and perpendicular to $\mathbf{a} = \langle a, b, c\rangle$ has equation
$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$
- How can we describe a 2D line living in 3D space?

Wed Sept 4: Lines and Planes, Cross Product (Sections 10.4 and 10.5)

- The line in 2D or the plane in 3D containing the point $\mathbf{r}_0$ and perpendicular to the vector $\mathbf{n}$ has the equation $\mathbf{n} \bullet (\mathbf{r} - \mathbf{r}_0) = 0$ or, equivalently, $\mathbf{n} \bullet \mathbf{r} = \mathbf{n} \bullet \mathbf{r}_0$.
- A line in 3D cannot be expressed with a single number equation. It can be expressed with a single "vector equation":
$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}.$$
This describes a particle that starts at $\mathbf{r}_0 = (x_0, y_0, z_0)$ and travels with constant velocity $\mathbf{v} = \langle a, b, c\rangle$. The position of the particle at time t is $\mathbf{r}(t) = (x(t), y(t), z(t))$. The single vector equation becomes three "parametric equations":
$$\begin{cases} x(t) = x_0 + ta, \\ y(t) = y_0 + tb, \\ z(t) = z_0 + tc. \end{cases}$$
- By eliminating t we can also express this line via three "symmetric equations":
$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$
Geometrically, this describes the line as an intersection of three planes.
- The line $\mathbf{r}(t) = (1 - t)\mathbf{r}_0 + t\mathbf{r}_1$ travels from $\mathbf{r}_0 \to \mathbf{r}_1$ as t goes from $0 \to 1$.
- Example 10.5.5: Find an equation of the plane that contains the points $P(1, 3, 2)$, $Q(3, -1, 6)$ and $R(5, 2, 0)$. Hmm, we need a trick.
- Given 3D[1] vectors $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3\rangle$, we define the *cross product*:
$$\mathbf{a} \times \mathbf{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1\rangle.$$
Note that $\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$. The whole point of the definition is that $\mathbf{a} \times \mathbf{b}$ is simultaneously perpendicular to $\mathbf{a}$ and $\mathbf{b}$:
$$\mathbf{a} \bullet (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \bullet (\mathbf{a} \times \mathbf{b}) = 0.$$
- This allows us to complete Example 10.5.5.
- The algebra of the cross product is strange, so be careful (see Theorem 8 on page 568).

[1] The cross product is a special trick that only works in 3D.

- Geometrically we have
$$\begin{aligned}|\mathbf{a}\times\mathbf{b}| &= |\mathbf{a}||\mathbf{b}|\sin\theta \\ &= \mathbf{area}(\text{parallelogram determined by } \mathbf{a} \text{ and } \mathbf{b}) \\ &= 2\cdot\mathbf{area}(\text{triangle with sides } \mathbf{a} \text{ and } \mathbf{b})\end{aligned}$$
- Right hand rules: $\mathbf{i}\times\mathbf{j}=\mathbf{k}$, $\mathbf{j}\times\mathbf{k}=\mathbf{i}$ and $\mathbf{k}\times\mathbf{i}=\mathbf{j}$.

Mon Sept 9: Vector Functions and Space Curves (Section 10.7)

- HW2 Discussion (10.3.45, 10.4.13, 10.4.14, 10.4.43, 10.5.10, 10.5.40)
- We can think of a function $f:\mathbb{R}\to\mathbb{R}^2$ or $f:\mathbb{R}\to\mathbb{R}^3$ as a "parametrized curve in 2D or 3D space." For this reason we will denote the input variable by t (for time). Thus for each value of $t\in\mathbb{R}$ we obtain a vector $f(t)\in\mathbb{R}^2$ in the plane or a vector $f(t)\in\mathbb{R}^3$ in space.
- Since $f(t)$ is a vector we should use boldface: $\mathbf{f}(t)\in\mathbb{R}^3$. Then since the x,y,z-coordinates are also functions of time we will write:
$$\mathbf{f}(t)=\langle x(t),y(t),z(t)\rangle = x(t)\mathbf{i}+y(t)\mathbf{j}+z(t)\mathbf{k}.$$
Actually, our textbook likes to use $\mathbf{r}$ instead of $\mathbf{f}$ for vector functions:[2]
$$\mathbf{r}(t)=\langle x(t),y(t),z(t)\rangle = x(t)\mathbf{i}+y(t)\mathbf{j}+z(t)\mathbf{k}.$$
- The *velocity* of the path $\mathbf{r}(t)$ at time t is the vector defined by differentiating each coordinate with respect to t:
$$\text{(instantaneous) velocity at time } t = \mathbf{r}'(t)=\langle x'(t),y'(t),z'(t)\rangle.$$
The *speed* of the path $\mathbf{r}(t)$ at time t is the length of the velocity vector:
$$\text{(instantaneous) speed at time } t = |\mathbf{r}'(t)| = \sqrt{[x'(t)]^2+[y'(t)]^2+[z'(t)]^2}.$$
Observe that the function $\mathbf{r}(t)/|\mathbf{r}'(t)|$ represents a path with a **constant speed**[3] of 1 unit per second.
- Examples: Parametrized circle $\mathbf{r}(t)=\langle\cos t,\sin t\rangle$ in the xy-plane. (It turns counterclockwise.) Parametrized circle $\mathbf{r}(t)=\langle 0,\sin t,\cos t\rangle$ in the yz-plane. (Which way does it turn?)
- Example 10.7.6: Find a parametrization for the curve defined implicitly by $x^2+y^2=1$ and $y+z=2$. [Infinitely many correct answers.] Reparametrize the curve so that it has constant speed.
- Differentiation of vector functions works just like differentiation of number functions. See the rules on page 593.

Wed Sept 11: Space Curves, Arc Length (Sections 10.7 and 10.8)

- Example 10.7.10 (parametrized helix)
- The tangent line at a point on a curve.
- Differentiation rules for vector functions (page 593)
- Application: If you travel on a circle or on the surface of a sphere then your velocity is always perpendicular to the radius.
- What kind of curve has constant velocity? Integration of vector functions.

[2] I guess the $\mathbf{r}$ stands for "radius," but I don't see why that's a good notation.

[3] This does not imply constant velocity. For example, the circle $\langle\cos t,\sin t\rangle$ has constant speed but does not have constant velocity because it it always changing direction. A curve with constant velocity is a straight line.

- Constant velocity $\mathbf{v}(t) = \langle a, b, c\rangle$ means
$$\mathbf{r}(t) = \int \mathbf{v}(t)\,dt = \left\langle \int a\,dt, \int b\,dt, \int c\,dt \right\rangle = \langle at+d, bt+e, ct+f\rangle$$
for some arbitrary constants d, e, f. This is the straight line starting at the point $\langle d, e, f\rangle$ and then moving in the direction of $\langle a, b, c\rangle$:
$$\mathbf{r}(t) = \langle d, e, f\rangle + t\langle a, b, c\rangle.$$
- Example 10.7.5 (parametrized line)
- How to compute the length of a curve? Remember that:
$$(\text{distance}) = \int (\text{speed})$$
- Example 10.8.1 (length of helix)
- Example 10.8.2 (parametrize helix by arc length)
- Why bother? Curvature: The rate of change of direction with respect to arc length.

Mon Sept 16: Curvature and Acceleration (Sections 10.8 and 10.9)

- Physics example: Exercise 10.9.11
- What is curvature? Recall from Calc I that $f''(x) < 0$ means "curving down" and $f''(x) > 0$ means "curving up." How can we extend this idea to curves in 3D?
- Unit speed curves (also called parametrizing by arclength). Exercise 10.8.10
- If $|\mathbf{r}'(t)| = |\mathbf{v}(t)| = 1$ for all t then $\kappa(t) = |\mathbf{a}(t)|$ is called the *curvature*.
- Curvature of a straight line is zero.
- Curvature of a circle of radius a:
$$\begin{aligned} \mathbf{r}(t) &= \langle a\cos(t/a), a\sin(t/a)\rangle \\ \mathbf{v}(t) &= \langle -\sin(t/a), \cos(t/a)\rangle, && \text{(unit speed)} \\ \mathbf{a}(t) &= \langle -\cos(t/a)/a, -\sin(t/a)/a\rangle, \end{aligned}$$
hence $\kappa = |\mathbf{a}(t)| = 1/a$. The circle gets flatter as $a \to \infty$.
- It is sometimes difficult to parametrize a curve by arc length, so we have a shortcut formula for curvature. Let $\mathbf{T}(t) = \mathbf{r}'(t)/|\mathbf{r}'(t)|$ be the unit tangent vector. Then
$$\kappa(t) = |\mathbf{T}'(t)|/|\mathbf{r}'(t)|.$$
In 3D we have
$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$
- Application: Curvature of the graph of a function f. Trick: Let $\mathbf{r}(t) = \langle t, f(t), 0\rangle$ so the graph is in the xy-plane. Then $\mathbf{r}'(t) = \langle 1, f'(t), 0\rangle$, $\mathbf{r}''(t) = \langle 0, f''(t), 0\rangle$ and
$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \langle 1, f'(t), 0\rangle \times \langle 0, f''(t), 0\rangle = \langle 0, 0, f''(t)\rangle.$$
It follows that
$$\kappa(t) = \frac{|f''(t)|}{\sqrt{1 + [f'(t)]^2}}.$$
- Actually the curvature is not just a number, it is a circle. Locally the curve is well approximated by the circle through $\mathbf{r}(t)$ with center at $\mathbf{r}(t) + \mathbf{N}(t)/\kappa(t)$, where $\mathbf{N}(t) = \mathbf{T}'(t)/|\mathbf{T}'(t)|$. (This is called the *osculating circle* at the point $\mathbf{r}(t)$.)

Wed Sept 18: Intro to Partial Derivatives (Sections 11.1, 11.3, 11.4)

- HW3 Discussion (10.7.5-6-9-24 with WolframAlpha,10.7.81,10.8.3,10.8.21,10.9.12)
- We find the linear approximation to the function $f(x)$ near $x = a$ by computing the tangent line to the graph at the point $(a, f(a))$:

$$f(x) \approx f(a) + (x - a)f'(a) \quad \text{when } x \approx a.$$

- We find the linear approximation to the function $\mathbf{r}(t)$ near $t = a$ by computing the tangent line to the curve at the point $\mathbf{r}(a)$:

$$\mathbf{r}(t) \approx \mathbf{r}(a) + (t - a)\mathbf{r}'(a) \quad \text{when } t \approx a.$$

- How can we compute the linear approximation to a function $f(x, y)$ with two inputs?
- Idea of the tangent plane.
- Idea of partial derivatives.
- The linear approximation is

$$f(x, y) \approx f(a, b) + (x - a)f_x(a, b) + (y - b)f_y(a, b).$$

We'll see why next time.

Mon Sept 23: Level Sets, Tangent Planes (Sections 11.1, 11.3, 11.4)

- A function $f : \mathbb{R}^2 \to \mathbb{R}$ with two real inputs and one real output can be visualized as a surface living in xyz-space. This surface is called the *graph* of the function. It is the set of all points (x, y, z) where $z = f(x, y)$.
- These are hard to draw. Sometimes it is easier to draw a bunch of *level sets* $f(x, y) = k$ for various constants k. This is like a topographic map of the surface.
- Example: The graph of $z = \sqrt{1 - x^2 - y^2}$ is a hemisphere.
- Example: The graph of $z = xy$ is a saddle ("hyperbolic paraboloid").
- What about the graph of $z = x^2 - y^2$? This is the same saddle, just rotated by 45°.
- If the function $f(x, y)$ is smooth ("differentiable") near the point $(x, y) = (a, b)$ then there exists a unique *tangent plane* to the surface at the point $(a, b, f(a, b))$. We want to find the equation of this plane.
- For this we need to find any two vectors in the plane. Easy: Consider the paths $\mathbf{r}_x(t) = \langle t, b, f(t, b)\rangle$ and $\mathbf{r}_y(t) = \langle a, t, f(a, t)\rangle$ which lie in the surface and pass through the point $(a, b, f(a, b))$. Then $\mathbf{r}_x'(a)$ and $\mathbf{r}_y'(b)$ are two vectors in the tangent plane.
- Compute:

$$\begin{aligned}
\mathbf{r}_x'(t) &= \langle 1, 0, \frac{d}{dt} f(t, b)\rangle, \\
\mathbf{r}_x'(a) &= \langle 1, 0, \frac{d}{dt} f(t, b) \text{ evaluated at } t = a\rangle, \\
\mathbf{r}_y'(t) &= \langle 0, 1, \frac{d}{dt} f(a, t)\rangle, \\
\mathbf{r}_y'(b) &= \langle 0, 1, \frac{d}{dt} f(a, t) \text{ evaluated at } t = b\rangle.
\end{aligned}$$

- Here's a more standard notation:

$$\begin{aligned}
f_x(a, b) &= \frac{d}{dt} f(t, b) \text{ evaluated at } t = a, \\
f_y(a, b) &= \frac{d}{dt} f(a, t) \text{ evaluated at } t = b.
\end{aligned}$$

- The tangent plane through $(a, b, f(a, b))$ contains the tangent vectors $\langle 1, 0, f_x(a, b)\rangle$ and $\langle 0, 1, f_y(a, b)\rangle$. Thus it has normal vector
$$\langle 0, 1, f_y(a, b)\rangle \times \langle 1, 0, f_x(a, b)\rangle = \langle f_x(a, b), f_y(a, b), -1\rangle.$$
Thus the equation of the tangent plane is
$$(x - a)f_x(a, b) + (y - b)f_y(a, b) + (z - f(a, b))(-1) = 0$$
- Another way to say this:
$$z - f(a, b) = (x - a)f_x(a, b) + (y - b)f_y(a, b),$$
$$\Delta z = f_x(a, b)\Delta x + f_y(a, b)\Delta y.$$
- Examples 11.4.1 and 11.4.3.
- Slightly different notation: If $z = f(x, y)$ then a tiny change in z is related to tiny changes in x and y as follows:
$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy.$$

Wed Sept 25 and Mon Sept 30 : Discuss HW4, Review for Exam 1

- HW4 Discussion (11.1.25-31-33 on GeoGebra, 11.3.12, 1.3.57,1.3.64)
- Discuss Practice Problems

Wed Oct 2: Exam 1

Mon Oct 7: Maximia and Minima (Section 11.7)

- Discuss Exam 1 solutions.
- Zeroth order (constant) approximation of $f(x, y)$ near (a, b):
$$f(x, y) \approx f(a, b).$$
- First order (linear) approximation of $f(x, y)$ near (a, b):
$$f(x, y) \approx f(a, b) + (x - a)f_x(a, b) + (y - b)f_y(a, b).$$
- Second order (quadratic) approximation of $f(x, y)$ near (a, b):
$$\begin{aligned} f(x, y) \approx{}& f(a, b) + (x - a)f_x(a, b) + (y - b)f_y(a, b) \\ &+ \frac{1}{2}\left[(x - a)^2 f_{xx}(a, b) + 2(x - a)(y - b)f_{xy}(a, b) + (y - b)^2 f_{yy}(a, b)\right]. \end{aligned}$$
- We say (a, b) is a *critical point* of $f(x, y)$ when $f_x(a, b) = 0$ and $f_y(a, b) = 0$, i.e., when the tangent plane is horizontal. In this case the point $(a, b, f(a, b))$ could be a local max, a local min, a saddle point, a cylinder or a flat plane.
- To distinguish these cases we define the *Hessian*:
$$D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - f_{xy}(a, b)^2.$$
- Let (a, b) be a critical point of $f(x, y)$. Then:
  - $D(a, b) > 0$ and $f_{xx}(a, b) > 0$ means local minimum.
  - $D(a, b) > 0$ and $f_{xx}(a, b) < 0$ means local maximum.
  - $D(a, b) < 0$ means saddle point.
  - If $D(a, b) = 0$ then the graph is too flat to tell anything from the second derivatives. One would need to compute higher derivatives.
- For *absolute extrema* on a domain we must also look at values on the boundary.
- Examples 11.7: 1, 2, 3, 6.

Wed Oct 9: Chain Rule, Gradient Vectors (Sections 11.5, 11.6)

- HW5 Discussion (11.7.28 on GeoGebra, 11.7.31, 11.7.43)
- In order to compute the tangent plane to the graph of $f(x, y)$ we needed compute the slope in the x direction and the slope in the y direction:
$$\text{slope in } x \text{ direction} = D_x f = \frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x+h, y) - f(x, y)}{h},$$
$$\text{slope in } y \text{ direction} = D_y f = \frac{\partial f}{\partial y} = \lim_{h\to 0} \frac{f(x, y+h) - f(x, y)}{h}.$$
- More generally, for any unit vector $\mathbf{u} = \langle a, b\rangle$ we can compute the slope in the $\mathbf{u}$ direction. This is called a *directional derivative*:
$$D_{\mathbf{u}} f := \text{slope in the } \mathbf{u} \text{ direction} = \lim_{h\to 0} \frac{f(x+ta, y+tb) - f(x, y)}{h}.$$
Indeed, let $\mathbf{r}(t) = \langle x+ta, y+tb, f(x+ta, y+tb)\rangle$ be the path in the surface $z = f(x, y)$ moving in the direction of $\mathbf{u} = \langle a, b\rangle$. Then the velocity vector is $\mathbf{r}'(t) = \langle a, b, D_{\mathbf{u}} f\rangle$, which has
$$\text{slope in the } \mathbf{u} \text{ direction} = \frac{\text{rise}}{\text{run}} = \frac{D_{\mathbf{u}} f}{\sqrt{a^2+b^2}} = D_{\mathbf{u}} f.$$
- There is shortcut formula to compute the directional derivative. If $z = f(x, y)$ is a function of x and y and if $x(t)$ and $y(t)$ are functions of t then the *multivariable chain rule* says that
$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$
It follows from this that
$$D_{\mathbf{u}} f = \nabla f \bullet \mathbf{u} = \langle f_x, f_y\rangle \bullet \langle a, b\rangle = f_x a + f_y b,$$
where $\nabla f = \langle f_x, f_y\rangle$ is called the *gradient vector*.
- Note that the slope $D_{\mathbf{u}} f = \nabla f \bullet \mathbf{u}$ is maximized when the vector $\mathbf{u}$ points in the same direction as the gradient ∇f. In other words: The gradient vector ∇f tells us the *direction of steepest ascent*.
- All of these concepts apply without change to functions of three or more variables.

Mon Oct 14: Implicit Differentiation, Lagrange Multipliers (Sections 11.5, 11.8)

- Let $u(x_1, \ldots, x_n)$ be a function of n variables and suppose that each variable x_i is a function of t. Then we can also view u as a function of t and the *multivariable chain rule* says that
$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t}.$$
I wrote everything in terms of partial derivatives because there may be other variables floating around; we don't care.
- Why is the chain rule true? Let $z = f(x, y)$ and recall that the tangent plane at $(x, y) = (a, b)$ has equation
$$z - f(a, b) = \frac{\partial z}{\partial x}(x-a) + \frac{\partial z}{\partial y}(y-b).$$

Now let $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ be any path in the surface with $\mathbf{r}(0) = \langle a, b, f(a, b)\rangle$. Then we observe that the point $\langle x, y, z\rangle = \langle a, b, f(a, b)\rangle + \mathbf{r}'(0) = \langle a + dx/dt, b + dy/dt, f(a, b) + dz/dt\rangle$ is in the tangent plane, hence

$$\begin{aligned} z - f(a,b) &= \frac{\partial z}{\partial x}(x-a) + \frac{\partial z}{\partial y}(y-b) \\ [f(a,b) + (dz/dt)] - f(a,b) &= \frac{\partial z}{\partial x}([a + (dx/dt)] - a) + \frac{\partial z}{\partial y}([b + (dy/dt)] - b) \\ \frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}. \end{aligned}$$

It's all about the tangent plane. The same idea works in higher dimensions.

- The chain rule allows us to compute tangent planes to *implicitly defined surfaces*. Suppose that a surface is implicitly defined by $F(x, y, z) = k$ and let $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ be a path in the surface. Then the chain rule says

$$\begin{aligned} F(x(t), y(t), z(t)) &= k \\ \frac{d}{dt}F(x(t), y(t), z(t)) &= 0 \\ \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt} &= 0. \end{aligned}$$

  In other words, the *gradient vector* $\nabla F = \langle F_x, F_y, F_z\rangle$ is perpendicular to every tangent vector $\mathbf{r}'(t) = \langle dx/dt, dy/dt, dz/dt\rangle$ at every point. Hence the equation of the tangent plane at $(x, y, z) = (a, b, c)$ is

$$F_x(a,b,c)(x-a) + F_y(a,b,c)(y-b) + F_z(a,b,c)(z-c) = 0.$$

- *Method of Lagrange Multipliers.* Suppose we want to maximize/minimize a function $f(x, y, z)$ subject to a *constraint* $g(x, y, z) = k$. For geometric reasons, the function will be maximized when the gradient vectors $\nabla f = \langle f_x, f_y, f_z\rangle$ and $\nabla g = \langle g_x, g_y, g_z\rangle$ point in the same direction:

$$\langle f_x, f_y, f_z\rangle = \lambda\langle g_x, g_y, g_z\rangle \quad \text{for some nonzero real number } \lambda.$$

  Thus the critical points are determined by the following system of four equations in four unknowns (x, y, z, λ):

$$\begin{cases} f_x(x,y,z) &= \lambda g_x(x,y,z), \\ f_y(x,y,z) &= \lambda g_y(x,y,z), \\ f_z(x,y,z) &= \lambda g_z(x,y,z), \\ g(x,y,z) &= k. \end{cases}$$

  Sometimes (for example, in textbook problems) these equations are solvable by hand. Sometimes this method is much faster than our previous methods.
- Examples: 11.5.5, 11.5.6, 11.6.7
- Exercises: 11.7.28 and 11.7.43 (revisit using Lagrange multipliers)

Wed Oct 16: Chapter 11 Review

- HW6 Discussion (11.5.10, 11.6.29, 11.6.49, 11.8.3, 11.8.5 with GeoGebra)
- Review of Chapter 11

Mon Oct 21: Intro to Double Integrals (Sections 12.1 and 12.2)

- In Chapter 11 we differentiated functions of type $f(x,y)$ and $f(x,y,z)$. In Chapter 12 we will integrate these functions. As before, the basic calculations are the same as in Calc I and II, but setting up the problems is trickier.
- To integrate $f(x,y)$ over a rectangle $(x,y) \in R = [a,b] \times [c,d]$ we can compute the integrals in either order. This is called *Fubini's theorem*:
$$\iint_R f(x,y)\,dx\,dy = \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy = \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx.$$
The result is the "signed volume" between the surface $z = f(x,y)$ and the x,y-plane. (Volume below the x,y-plane is recorded as negative.)
- Sometimes we write $dx\,dy = dA$ (for area) to emphasize that the order doesn't matter:
$$\iint_R f(x,y)\,dA = \iint_R f(x,y)\,dx\,dy = \iint_R f(x,y)\,dy, dx.$$
Think of $f(x,y)\,dA$ as the volume of a skinny rectangular box with height $f(x,y)$ and base area dA. The integral adds up all the little volumes.
- We can also integrate over a general region D in the x,y-plane. The only difficulty is to parametrize the region. Example 12.2.2: Compute the volume $V =$
$$\iint_D (x^2+y^2)\,dA, \text{ where } D \text{ is the region between the curves } y = 2x \text{ and } y = x^2.$$
Solution 1: By thinking vertically, $D = \{(x,y) : 0 \le x \le 2, x^2 \le y \le 2x\}$, we get
$$V = \iint_D (x^2+y^2)\,dA = \int_0^2 \left(\int_{x^2}^{2x} (x^2+y^2)\,dy \right) dx.$$
Solution 2: By thinking horizontally, $D = \{(x,y) : 0 \le y \le 4, y/2 \le x \le \sqrt{y}\}$, we get
$$V = \iint_D (x^2+y^2)\,dA = \int_0^4 \left(\int_{y/2}^{\sqrt{y}} (x^2+y^2)\,dx \right) dy.$$
Check that you get the same volume $V = 216/35$ each time.
- Sometimes one of the two methods (vertical or horizontal) is easier. Sometimes both methods are too hard. We will learn a few tricks, but you should be aware that integration in general is a task best left to computers.
- Here's a trick: Break the region D into manageable pieces D_1 and D_2. Then we have
$$\iint_D f(x,y)\,dA = \iint_{D_1} f(x,y)\,dA + \iint_{D_2} f(x,y)\,dA.$$

Wed Oct 23: Double Integrals in Polar Coordinates (Section 12.3)

- Polar coordinates are defined by $x = r\cos\theta$ and $y = r\sin\theta$, hence also $x^2+y^2 = r^2$. The element of area in polar coordinates is a bit tricky:
$$dA = dx\,dy = r\,dr\,d\theta.$$
There are various ways to see this. The easiest way is to draw the picture of the region defined by $[r, r+\Delta r]$ and $[\theta, \theta+\Delta\theta]$. It looks like a slightly bendy rectangle with dimensions Δr and $r\Delta\theta$, hence
$$\Delta A \approx r\,\Delta r\,\Delta\theta.$$
- Polar coordinates are helpful when we integrate over a domain D that is a circle, annulus, ellipse, or some kind of round shape.

- Example: Area of a circle. Let $D = \{(x,y) : x^2 + y^2 \leq R^2\}$ and compute
$$\text{(area of circle)} = \iint_D 1\,dA = \iint_D 1\,r\,dr\,d\theta = \int_0^{2\pi}\int_0^R r\,dr\,d\theta.$$
This would be much harder in Cartesian coordinates.
- Example: Volume of a sphere. Consider $f(x,y) = \sqrt{R^2 - x^2 - y^2}$ and $D = \{(x,y) : x^2 + y^2 \leq R^2\}$. Then
$$\frac{1}{2}\text{(volume of sphere)} = \iint_D f(x,y)\,dA = \int_0^{2\pi}\int_0^R \sqrt{R^2 - r^2}\cdot r\,dr\,d\theta$$
- Examples 12.3.1 and 12.3.3.

Mon Oct 28: Center of Mass (Sections 7.6 and 12.4)

- Archimedes' Law of the Lever (see page 410): $m_1(\bar{x} - x_1) = m_2(x_2 - \bar{x})$.
- Recall from Section 7.6: The *center of mass* for a collection of point particles with positions $(x_1, y_1), \ldots, (x_n, y_n)$ and masses $m_1, \ldots, m_n$ is
$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m}\right) = \left(\frac{\sum_i m_i x_i}{\sum_i m_i}, \frac{\sum_i m_i y_i}{\sum_i m_i}\right).$$
Remark: Take note of the funny notation $M_y = \sum_i x_i m_i$. This is called the *moment about the y-axis*; to compute it we sum over the x-coordinates, which are **perpendicular** to the y-axis. This notational convention comes from physics.
- Now we will replace the point masses by a *density function* $\rho(x,y)$ and we will replace sums by integrals:
$$\begin{aligned} m &= \sum_i m_i & &\longrightarrow & m &= \iint \rho(x,y)\,dA, \\ M_y &= \sum_i x_i m_i & &\longrightarrow & M_y &= \iint x\rho(x,y)\,dA, \\ M_x &= \sum_i y_i m_i & &\longrightarrow & M_x &= \iint y\rho(x,y)\,dA. \end{aligned}$$
The most interesting case is when $\rho(x,y) = k$ for some constant.
- Example: Center of mass of semicircle $D = \{(x,y) : -R \leq x \leq R, 0 \leq y \leq \sqrt{R^2 - x^2}\}$ with constant density $\rho(x,y) = k$. We already know $m = \frac{\pi R^2}{2}k$ is just the area times k. The *moment about the y-axis* is
$$\begin{aligned} M_y &= \iint x\rho(x,y)\,dA \\ &= \int_{-R}^{R}\left(\int_0^{\sqrt{R^2-x^2}} xk\,dy\right)dx \\ &= k\int_{-R}^{R} x\sqrt{R^2 - x^2}\,dx \\ &= 0 \quad \text{because the integrand is odd.} \end{aligned}$$

We could also say that $\bar{x} = 0$ because the semicircle is symmetric about the y-axis. The *moment about the x-axis* is

$$\begin{aligned} M_x &= \iint y\rho(x,y)\,dA \\ &= \int_{-R}^{R} \left(\int_0^{\sqrt{R^2-x^2}} yk\,dy \right) dx \\ &= k\int_{-R}^{R} \frac{R^2-x^2}{2}\,dx \\ &= \frac{k}{2}\left[R^2x - \frac{x^3}{3} \right]_{-R}^{R} = k\left[R^3 - \frac{R^3}{3} \right] = \frac{k2R^3}{3}. \end{aligned}$$

Thus the center of mass is

$$(\bar{x},\bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m} \right) = \left(\frac{0}{k\pi R^2/2}, \frac{k2R^3/3}{k\pi R^2/2} \right) = \left(0, \frac{4R}{3\pi} \right).$$

Compare to Example 7.6.8 on page 413.

- Example 12.4.3: Same domain but now with density $\rho(x,y) = K\sqrt{x^2+y^2}$ for some constant K. This time it's easier to use polar coordinates.
- WE SKIPPED MOMENTS OF INERTIA (i.e., SECOND MOMENTS)

Wed Oct 30: Intro to Triple Integrals (Section 12.5)

- HW7 Discussion (12.3.23, 12.3.26, 12.4.11, 12.3.17)
- Everything we have done for functions $f(x,y)$ generalizes to functions $f(x,y,z)$. First we integrate over a rectangular box $B = [a,b]\times[c,d]\times[r,s]$:

$$\iiint_B f(x,y,z)\,dV = \iiint_B f(x,y,z)\,dx\,dy\,dz = \int_r^s \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz.$$

  If you want you can think of this triple integral as a "hypervolume" in 4D space (though in applications it usually has a different interpretation, such as mass). *Fubini's Theorem* again says that we can integrate the variables in any order.
- We can also integrate over a general solid region E:

$$\iiint_E f(x,y,z)\,dV.$$

  There are no new ideas here; it just becomes trickier to parametrize the domain. Next time we will learn some tricks (cylindrical and spherical coordinates) to make certain kinds of domains easier to deal with.
- Come to class and see!

Mon Nov 4: Cylindrical and Spherical Coordinates (Sections 12.6 and 12.7)

- Volume of a Tetrahedron (Based on Example 12.5.2)
- In *cylindrical coordinates* we replace x,y,z by r,θ,z (i.e., we keep z the same), where

$$\left\{ \begin{array}{lcl} x &=& r\cos\theta \\ y &=& r\sin\theta \end{array} \right\} \iff \left\{ \begin{array}{lcl} r &=& \sqrt{x^2+y^2} \\ \theta &=& \arctan(y/x) \end{array} \right\}.$$

  It follows as with polar coordinates that $dx\,dy = r\,dr\,d\theta$, and hence

$$dV = dx\,dy\,dz = r\,dr\,d\theta\,dz.$$

- Example: Use cylindrical coordinates to compute the volume of ice cream between the cone $z^2 = x^2 + y^2$ and the sphere $x^2 + y^2 + z^2 = 1$. Answer:
$$\iiint_{\text{ice cream}} dV = \int_0^{2\pi} \int_0^{1/\sqrt{2}} \int_r^{\sqrt{1-r^2}} r\, dr\, d\theta\, dz = \frac{2\pi}{3}\left(1 - \frac{1}{\sqrt{2}}\right) \approx 0.613$$
(Compare to Exercise 12.3.17 from HW7.)
- Maybe there is an easier way to compute this volume? In *spherical coordinates* we replace x, y, z by ρ, θ, ϕ, where
$$\left\{\begin{array}{rcl} x &=& \rho\sin\phi\cos\theta \\ y &=& \rho\sin\phi\sin\theta \\ z &=& \rho\cos\phi \end{array}\right\} \iff \left\{\begin{array}{rcl} \rho &=& \sqrt{x^2+y^2+z^2}, \\ \theta &=& \arctan(y/x) \\ \phi &=& \arccos(z/\sqrt{x^2+y^2+z^2}) \end{array}\right\}.$$
This time it's trickier to compute the element of volume. The region of space defined by $[\rho, \rho+\Delta\rho]$, $[\theta, \theta+\Delta\theta]$ and $[\phi, \phi+\Delta\phi]$ looks like a curvy rectangular box with dimensions $\Delta\rho$, $\rho\Delta\phi$ and $\rho\sin\phi\Delta\theta$. It follows that
$$\Delta V \approx (\Delta\rho)(\rho\Delta\phi)(\rho\sin\phi\Delta\theta) = \rho^2\sin\phi\Delta\rho\Delta\phi\Delta\theta.$$
and hence $dV = \rho^2 \sin\phi\, d\rho, d\phi, d\theta$.
- Now it's easier to compute the volume of ice cream:
$$\iiint_{\text{ice cream}} dV = \int_0^{2\pi} \int_0^{\pi/4} \int_0^1 \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta = \frac{2\pi}{3}\left(1 - \frac{1}{\sqrt{2}}\right) \approx 0.613$$
- Might as well compute the volume of a whole sphere:
$$\iiint_{\text{sphere of radius } R} dV = \int_0^{2\pi} \int_0^{\pi} \int_0^R \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta = \frac{4\pi R^3}{3}.$$

Wed Nov 6: Out Sick

Mon Nov 11: Intro to Vector Fields (Sections 13.1 and 13.5)

- In Chapter 10 dealt with differentiation and integration of functions $f : \mathbb{R} \to \mathbb{R}^n$ where $n = 2$ or $n = 3$ (curves in the plane and in space). Chapters 11 and 12 dealt with differentiation and integration of functions $f : \mathbb{R}^n \to \mathbb{R}$ where $n = 2$ or $n = 3$. Chapter 13 deals with functions of the form $f : \mathbb{R}^n \to \mathbb{R}^n$ for $n = 2$ or $n = 3$, i.e., with *vector fields* in the plane or in space.
- Let $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ be a function with two inputs and two outputs.[4] That is, for each point (x, y) we have a vector $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y)\rangle = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$. We can think of the coordinates $P(x, y)$ and $Q(x, y)$ as functions $\mathbb{R}^2 \to \mathbb{R}$, hence we can apply to them the techniques of Chapters 11 and 12.
- A function $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ or $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ is called a *vector field* because it associates a vector $\mathbf{F}(\mathbf{x})$ to each point $\mathbf{x}$. You can think of the vector $\mathbf{F}(\mathbf{x})$ as the velocity of a tiny particle at position $\mathbf{x}$ (in which case $\mathbf{F}$ is a *velocity field*), or you could think of it as the force felt by a tiny particle at $\mathbf{x}$ (in which case $\mathbf{F}$ is a *force field*).
- Problem: Given a velocity field $\mathbf{F}$ and a point $\mathbf{x}_0$, find the curve $\mathbf{r}(t)$ such that $\mathbf{r}(0) = \mathbf{x}_0$ and $\mathbf{r}'(t) = F(\mathbf{r}(t))$ for all times t. This curve $\mathbf{r}(t)$ is called the *flow* of the field. (Drop a cork in the water and it will follow this curve.)

[4] We use boldface because the output is a vector.

- One natural way to create a vector field is to start with a *scalar function* $f : \mathbb{R}^2 \to \mathbb{R}$ or $f : \mathbb{R}^3 \to \mathbb{R}$ and to consider the *gradient vector* at each point:
$$\mathbf{F}(x, y, z) = \nabla f(x, y, z) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z)\rangle.$$
The flow of this *gradient field* takes you in the direction of maximum increase of the function f. You should follow this flow if you want to maximize f.
- Example: Consider the scalar function $f(x, y) = x^2/2 + y^2$, with gradient vector field $\mathbf{F}(x, y) = \nabla f = \langle f_x, f_y\rangle = \langle x, 2y\rangle$. Each level set of this function is an ellipse $x^2/2 + y^2 = k$. Recall that the gradient vector ∇f is always perpendicular to the level set. If $\mathbf{r}(t) = \langle x(t), y(t)\rangle$ is the flow line through the point $\mathbf{r}(0) = (x_0, y_0)$ then we must have $\mathbf{r}'(t) = \langle x'(t), y'(t)\rangle = \mathbf{F}(\mathbf{r}(t)) = \langle x(t), 2y(t)\rangle$. In other words, we have the following system of *differential equations* and *initial conditions*:
$$\left\{\begin{array}{rcr} x'(t) & = & x(t) \\ y'(t) & = & 2y(t) \end{array}\right\} \quad \text{and} \quad \left\{\begin{array}{rcr} x(0) & = & x_0 \\ y(0) & = & y_0 \end{array}\right\}$$
This system has a unique solution, which is a parabola:
$$\mathbf{r}(t) = \langle x(t), y(t)\rangle = \langle x_0 e^t, y_0 e^{2t}\rangle.$$
You should follow this curve if you want to optimize the function f.
- Finding the gradient flow is hard because it is a form of integration. In practice it is done by computers.
- You can think of the scalar function f as some kind of *antiderivative* of the vector function $\mathbf{F} = \nabla f$. Warning: Not every vector field has an antiderivative. The fields that do are called *conservative*:

  "$\mathbf{F}$ is a conservative field" means "$\mathbf{F} = \nabla f$ for some scalar function f."
- Why do we call $\mathbf{F} = \nabla f$ conservative? What does it conserve? Answer: Energy. If $\mathbf{F}$ is a force field then Newton's 2nd Law says a particle of mass m will follow a curve $\mathbf{r}(t)$ satisfying $\mathbf{F}(\mathbf{r}(t)) = m\mathbf{r}''(t)$ for all times t. We call $K(t) = \frac{1}{2}m|\mathbf{r}'(t)|^2$ the *kinetic energy* of the particle. If $\mathbf{F} = -\nabla P$ for some scalar function $P : \mathbb{R}^3 \to \mathbb{R}$ then we call $P(t) := P(\mathbf{r}(t))$ the *potential energy* of the particle. In this case one can show that the total energy $E(t) = K(t) + P(t)$ is **conserved**. That is, we have $\frac{d}{dt}E(t) = 0$.
- Examples of conservative forces: gravity, elastic, electrostatic, magnetic.
- Examples of non-conservative forces: friction.

Wed Nov 13: Survey of Vector Calculus (Sections 13.1 and 13.5, etc.)

- Last time we discussed mostly old concepts in the new language of vector fields. This time we will discuss some genuinely new properties of vector fields: the *curl* and the *divergence* of a field.
- Let $\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$ be a vector field. Then the *divergence* of $\mathbf{F}$ is a scalar field defined by
$$\operatorname{div}\mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z},$$
and the *curl* of $\mathbf{F}$ is a vector field defined by
$$\operatorname{curl}\mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$$

- If we pretend that the abstract symbol $\nabla = \langle \partial/\partial x, \partial/\partial y, \partial/\partial z\rangle$ is a vector (it's not), then we have the following mnemonic notations
$$\operatorname{div}\mathbf{F} = \nabla \bullet \mathbf{F} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\rangle \bullet \langle P, Q, R\rangle$$
and
$$\operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix}.$$
- But what do these definitions mean? If we think of $\mathbf{F}(x, y, z)$ as the velocity field of an incompressible fluid, then the number $\nabla \bullet \mathbf{F}(x, y, z)$ represents the amount of fluid entering or leaving the system at the point (x, y, z). If $\nabla \bullet \mathbf{F} = 0$ then there is *conservation of mass*. The number $|\nabla \times \mathbf{F}(x, y, z)|$ represents the amount of rotation of the fluid at the point (x, y, z) and the vector $\nabla \times \mathbf{F}(x, y, z)$ is the direction of the rotation (right hand rule).
- We can think of a 2D vector field $\mathbf{F}(x, y)$ as a 3D vector field $\mathbf{F}(x, y, z)$ with $R(x, y, z) = 0$. Then the curl always points in the z-direction: $\nabla \times \mathbf{F} = (\partial Q/\partial x - \partial P/\partial y)\mathbf{k}$. This suggests that we should define the following 2D scalar version of the curl:
$$\text{“curl”}\,\mathbf{F} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}.$$
- Important Theorem: The vector field $\mathbf{F}$ is conservative if and only if it has zero curl:
$$\mathbf{F} = \nabla f \text{ for some } f \quad \Longleftrightarrow \quad \nabla \times \mathbf{F} = \mathbf{0} \text{ at all points.}$$
- *Green, Stokes and Divergence Theorems*: Note that the gradient ∇f, the divergence $\nabla \mathbf{F}$ and the curl $\nabla \times \mathbf{F}$ are all different kinds of derivatives. It makes sense that these concepts should be related to integration via certain higher dimensional versions of the "fundamental theorem of calculus." The ideas are straightforward. If you integrate the gradient ∇f along a curve $\mathbf{r}(t)$ from $t = a$ to $t = b$ then you get $f(\mathbf{r}(b)) - f(\mathbf{r}(a))$. (It follows that the path of integration doesn't matter; just the endpoints.) If you integrate the curl $\nabla \times \mathbf{F}$ over a surface with boundary then all the little rotations cancel and you are left with just the integral around the boundary curve. If you integrate the divergence $\nabla \bullet \mathbf{F}$ over a solid region then this is the same as integrating the field $\mathbf{F}$ over the surface of the region. (Idea: If fluid is leaking outward/inward across the boundary then there must be some source/sink on the inside of the region.)
- The computational details for integrating along curves and surfaces are tedious and we didn't have time to discuss them.

Mon Nov 18: Review for Exam 2

Wed Nov 20: Exam 2