1. Let $\mathbf{u} = \langle -1, 2, 0 \rangle$, $\mathbf{v} = \langle 2, 0, 2 \rangle$ and $\mathbf{w} = \langle 0, 3, 1 \rangle$

   (a) Compute the area of the parallelogram spanned by $\mathbf{u}$ and $\mathbf{v}$.

   > **Solution:** We have $Area = \|\mathbf{u} \times \mathbf{v}\| = \| \langle 4, 2, -4 \rangle \| = \sqrt{36} = 6$.

   (b) Compute $\cos(\theta)$ where θ is the angle between $\mathbf{v}$ and $\mathbf{w}$.

   > **Solution:**
   > $$\cos(\theta) = \frac{\mathbf{v} \cdot \mathbf{w}}{\|v\|\|w\|} = \frac{2}{\sqrt{8}\sqrt{10}} = \frac{\sqrt{5}}{10}$$

   (c) Give a parametric equation for the line perpendicular to $\mathbf{u}$ and $\mathbf{v}$ and passing through the point $(-1, 2, 3)$.

   > **Solution:** Using the solution to part (a), we have that the line has direction vector $\langle 4, 2, -4 \rangle$ or $\langle 2, 1, -2 \rangle$. So a vector parameterization is
   > $$\mathbf{r}(t) = \langle -1, 2, 3 \rangle + t \langle 2, 1, -2 \rangle .$$

2. Consider the curve $\mathcal{C}$ given by the parametrization

   $$\mathbf{r}(t) = \left\langle 2t, t^2, t \right\rangle \quad \text{for } 0 \leq t \leq 2$$

   (a) Find the unit tangent vector $\mathbf{T}(t)$ of $\mathbf{r}(t)$.

   > **Solution:** Observe $\mathbf{r}'(t) = \langle 2, 2t, 1 \rangle$. so that $\|\mathbf{r}'(t)\| = \sqrt{4t^2 + 5}$ and
   > $$\mathbf{T}(t) = \frac{1}{\sqrt{4t^2 + 5}} \langle 2, 2t, 1 \rangle$$

3. Calculate the following quantities if they exist. Otherwise, explain why they do not exist. Justify either response.

(a) For

$$f(x,y,z) = e^{\sqrt{x^2+y^2}} + e^{\sqrt{z^2+y^2}} + 2xy$$

compute

$$f_{xz}(x,y,z)$$

Solution: The first term has z partial equal to zero and the second has x partial equal to zero, so the answer is identically zero.

(b) For $f(x,y,z) = y\sin(x) - 4e^z$ find a unit vector pointing in the direction where f increases the fastest, starting at $(0,3,0)$.

Solution: The gradient of f is $\nabla f = \langle y\cos(x), \sin(x), -4e^z\rangle$ which, at $(0,3,0)$ evaluates to $\langle 3, 0, -4\rangle$. This points in the direction of greatest increase. To find the unit vector in that direction, we divide by the norm $= \sqrt{9+16} = 5$ to obtain

$$\left\langle \frac{3}{5}, 0, -\frac{4}{5} \right\rangle$$

(c) Find the equation for the tangent plane to the surface

$$z = x^2 - y^2$$

at the point $(2,-1,3)$.

Solution: This is a level surface for $0 = x^2 - y^2 - z = f(x,y,z)$ and the gradient of f is $\nabla f = \langle 2x, -2y, -1\rangle$ which evaluates to $\langle 4, 2, -1\rangle$ at $(2,-1,3)$. As this is a normal vector to the tangent plane, we have that the equation is $4x + 2y - z = 4\cdot 2 + 2\cdot(-1) + (-1)\cdot 3 = 3$.

4. Let

$$f(x,y) = x^2 + \cos y$$

and

$$\mathcal{D} = [-1,1] \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

(a) Find the critical points of $f(x,y)$ in the interior of $\mathcal{D}$.

Solution: Solving $\nabla f = \langle 2x, -\sin y\rangle = \langle 0, 0\rangle$ gives $x = 0$ and $y = 0$ as the only critical point of f.

(b) Describe the local behavior of $f(x, y)$ at the critical points found in part (a).

Solution: The Hessian of f has $2, -1$ along the diagonal and zeros off the diagonal. Thus $(0, 0)$ is a saddle point.

(c) Find the global maximum and minimum values of $f(x, y) = x^2 + \cos y$ on $\mathcal{D}$.

Solution: Examining the horizontal sides of $\mathcal{D}$ where $y = \pm\pi/2$, we obtain $\cos(y) = 0$ and $f(x, y)$ is the function x^2 on $[-1, 1]$. This has a minimum value of 0 and maximum of 1. On the vertical sides we have $f(x, y) = \cos(y) + 1$ on $[-\pi/2, \pi/2]$ which has a minimum of 1 and maximum of 2. Thus the global min (max) are 0 (2) respectively.

5. Let $\mathcal{W} = [0, \pi] \times [-1, 1] \times [2, 3]$. Evaluate the triple integral

$$\iiint_{\mathcal{W}} (3y^2 + z) \sin x \; \mathrm{d}V$$

Solution: This is a straightforward computation using Fubini and product of integrals

$$\begin{aligned}
\iiint_{\mathcal{W}} (3y^2 + z) \sin x \; \mathrm{d}V &= \left(\int_0^{\pi} \sin x \; \mathrm{d}x\right)\left(\int_{-1}^{1}\int_2^3 3y^2 + z \; \mathrm{d}z \; \mathrm{d}y\right), \\
&= 2\int_{-1}^{1} 3zy^2 + \frac{z^2}{2}\bigg|_2^3 \; \mathrm{d}y, \\
&= 2\int_{-1}^{1} 3y^2 + \frac{5}{2} \; \mathrm{d}y, \\
&= 2\left(y^3 + \frac{5}{2}y\right)\bigg|_{-1}^{1}, \\
&= 14.
\end{aligned}$$

6. Evaluate the following integrals.

(a) Let $\mathcal{D}$ be the upper half-disc

$$\mathcal{D} : x^2 + y^2 \leq 4, y \geq 0$$

in the plane. Evaluate

$$\iint_{\mathcal{D}} 2e^{x^2+y^2} \; \mathrm{d}A.$$

Solution: Change to polar coordinates to obtain

$$\begin{aligned} \iint_{\mathcal{D}} 2e^{x^2+y^2} \; \mathrm{d}x \; \mathrm{d}y &= \int_0^{\pi} \int_0^2 2re^{r^2} \; \mathrm{d}r \; \mathrm{d}\theta, \\ &= \int_0^{\pi} e^4 - 1 \; \mathrm{d}\theta, \\ &= \pi e^4 - \pi. \end{aligned}$$

(b) Let $\mathcal{D}$ be the region $0 \leq x \leq 1 - y^2$. Evaluate

$$\iint_{\mathcal{D}} 1 + y^2 \; \mathrm{d}A.$$

Solution:

$$\begin{aligned} \iint_{\mathcal{D}} 1 + y^2 \; \mathrm{d}A &= \int_{-1}^{1} \int_0^{1-y^2} 1 + y^2 \; \mathrm{d}x \; \mathrm{d}y, \\ &= \int_{-1}^{1} (1 + y^2)(1 - y^2) \; \mathrm{d}y, \\ &= \int_{-1}^{1} 1 - y^4 \; \mathrm{d}y, \\ &= y - \frac{y^5}{5} \Big|_{-1}^{1} = 2 - \frac{2}{5} = \frac{8}{5}. \end{aligned}$$

(c) Let $\mathcal{E}$ be the half ball $x^2 + y^2 + z^2 \leq 1$ with $0 \leq z$ and evaluate

$$\iiint_{\mathcal{W}} z^2 \; \mathrm{d}V.$$

Solution:

$$\begin{aligned}\iiint_{\mathcal{W}} z^2 \ \mathrm{d}V &= \int_0^{2\pi}\int_0^{\pi/2}\int_0^1 \rho^2\cos^2(\phi)\rho^2\sin(\phi)\ \mathrm{d}\rho\ \mathrm{d}\phi\ \mathrm{d}\theta,\\ &= 2\pi\left(\int_0^{\pi/2}\cos^2(\phi)\sin(\phi)\ \mathrm{d}\phi\right)\left(\int_0^1 \rho^4\ \mathrm{d}\rho\right),\\ &= \frac{2\pi}{5}\left(-\left.\frac{\cos^3(\phi)}{3}\right|_0^{\pi/2}\right),\\ &= \frac{2\pi}{15}.\end{aligned}$$

7. Let $\mathbf{u} = \langle 1, 1, 0\rangle$, $\mathbf{v} = \langle 1, 0, 1\rangle$ and $\mathbf{w} = \langle 0, 1, 1\rangle$.

   (a) Compute the volume of the parallelepiped spanned by $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$.

   Solution: We have $Volume = |(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}|$ and $\mathbf{u}\times\mathbf{v} = \langle 1, -1, -1\rangle$ so that $Volume = 2$.

   (b) Compute the angle θ between $\mathbf{v}$ and $\mathbf{w}$.

   Solution:

   $$\cos(\theta) = \frac{\mathbf{v}\cdot\mathbf{w}}{\|v\|\|w\|} = \frac{1}{\sqrt{2}\sqrt{2}} = \frac{1}{2}$$

   So that $\theta = \pi/3$

   (c) Give the equation for the plane parallel to $\mathbf{u}$ and $\mathbf{v}$ and passing through the origin.

   Solution: Using the solution to part (a), we have that $\mathbf{u}\times\mathbf{v} = \langle 1, -1, -1\rangle$ is a normal vector for the plane so that

   $$x - y - z = 0$$

   is the equation.

8. Consider the curve $\mathcal{C}$ given by the parametrization

$$\mathbf{r}(t) = \left\langle \sin(t), \cos(t), e^t\right\rangle \quad \text{for } 0 \leq t \leq \pi$$

(a) Find the speed of $\mathbf{r}(t)$ as a function of t.

Solution: The speed is $\|\mathbf{r}'(t)\| = \| \langle \cos(t), -\sin(t), e^t \rangle \|$. so that $\|\mathbf{r}'(t)\| = \sqrt{\cos^2 t + \sin^2 t + e^{2t}} = \sqrt{1 + e^{2t}}$.

9. Calculate the following quantities if they exist. Otherwise, explain why they do not exist. Justify either response.

(a) For

$$f(x, y, z) = \cos(z^2 - y^2) + y \sin(x)$$

compute

$$f_{xy}(x, y, z)$$

Solution: The first term has x partial equal to zero and the second has x partial equal to $y \cos(x)$. Taking the y partial then gives $\cos(x)$.

(b) For $f(x, y, z) = x + y^2 + z^3$ find the change in $f(x, y, z)$ as one moves in the direction of the unit vector $\mathbf{u} = \frac{\sqrt{3}}{3} \langle 1, 1, 1 \rangle$ starting at $(2, 0, -1)$.

Solution: This is the directional derivative $D_{\mathbf{u}} f(2, 0, -1)$. To compute, we evaluate the gradient $\nabla f = \langle 1, 2y, 3z^2 \rangle$ at $(2, 0, -1)$ to get $\nabla f(2, 0, -1) = \langle 1, 0, 3 \rangle$. We then take the dot product to get

$$D_{\mathbf{u}} f(2, 0, -1) = \nabla f(2, 0, -1) \cdot \mathbf{u} = \frac{4\sqrt{3}}{3}.$$

(c) Find the equation for the tangent plane to the surface

$$z = xy$$

at the point $(1, 2, 2)$.

Solution: We compute $f_x(1, 2) = 2$, $f_y(1, 2) = 1$ and $f(1, 2) = 2$ so that the linearization is $L(x, y) = 2 + 2(x - 1) + (y - 2) = 2x + y - 2$ giving the equation

$$z = 2x + y - 2$$

10. Let

$$f(x, y) = x^3 - 12x + y^2$$

and $\mathcal{D}$ be the square $[-3, 3] \times [-3, 3]$.

(a) Find the critical points of $f(x, y)$ in the interior of $\mathcal{D}$.

> **Solution:** Solving $\nabla f = \langle 3x^2 - 12, 2y \rangle = \langle 0, 0 \rangle$ gives $x = \pm 2$ and $y = 0$. Thus $(2, 0)$ and $(-2, 0)$ are the only critical points of f.

(b) Describe the local behavior of $f(x, y)$ at the critical points found in part (a).

> **Solution:** We have that $f_{xx} = 6x$, $f_{yy} = 2$ and $f_{xy} = 0$. So at $(2, 0)$, the discriminant is $D = 24$ and $f_{xx} > 0$ so that $(2, 0)$ is a local min while at $(-2, 0)$, $D = -24$ so that there is a saddle point.

(c) Find the maximum value of f on $\mathcal{D}$.

> **Solution:** Since there is only a local min and saddle in the interior of $\mathcal{D}$, the maximum must occur on the boundary. For $y = \pm 3$, we have $g(x) = f(x, \pm 3) = x^3 - 12x + 9$ and the critical points are again solutions to $3x^2 - 12 = 0$ which give $2, -2$. Checking the values at these points we have $f(2, \pm 3) = -7$ and $f(-2, \pm 3) = 25$. The values at the endpoints are $f(-3, \pm 3) = 18$ and $f(3, \pm 3) = 0$.
>
> For $x = 3$, we have $g(x) = y^2 - 9$ which clearly has a minimum of -9 at 0 and maxima at the endpoints (already computed above). Similarly, at $x = -3$, we have $y^2 + 9$ which again has a minimum of 9 at 0 and maxima at endpoints (already computed). Thus the global maximum is 25.

11. Let $\mathcal{W} = [0, 1] \times [-1, 0] \times [0, 2]$. Evaluate the triple integral

$$\iiint_{\mathcal{W}} (2x + z)e^y \, \mathrm{d}V$$

Solution: This is a straightforward computation using Fubini and product of integrals

$$
\begin{aligned}
\iiint_{\mathcal{W}} (2x+z)e^y \ \mathrm{d}V &= \left(\int_{-1}^{0} e^y \ \mathrm{d}y\right)\left(\int_0^2\int_0^1 2x+z \ \mathrm{d}x \ \mathrm{d}z\right), \\
&= (1-1/e)\int_0^2 x^2+xz\Big|_0^1 \ \mathrm{d}z, \\
&= (1-1/e)\int_0^2 1+z \ \mathrm{d}z, \\
&= (1-1/e)\left(z+z^2/2\Big|_0^2\right), \\
&= 4(1-1/e).
\end{aligned}
$$

12. Evaluate the following integrals.

(a) Let $\mathcal{D}$ be the region $x^2+y^2 \le 4$, $0 \le y$, $x \le 0$. Evaluate

$$\iint_{\mathcal{D}} 3x \ \mathrm{d}A.$$

Solution: Switching to polar coordinates, we have that $\mathcal{D}$ is the region $\pi/2 \le \theta \le \pi$ and $0 \le r \le 2$.

$$
\begin{aligned}
\iint_{\mathcal{D}} 3x \ \mathrm{d}A &= \int_{\pi/2}^{\pi}\int_0^2 3r^2\cos\theta \ \mathrm{d}r \ \mathrm{d}\theta, \\
&= \int_{\pi/2}^{\pi} r^3\Big|_0^2 \cos\theta \ \mathrm{d}\theta, \\
&= 8\int_{\pi/2}^{\pi} \cos\theta \ \mathrm{d}\theta, \\
&= 8 \sin\theta\Big|_{\pi/2}^{\pi} = -8.
\end{aligned}
$$

(b) Let $\mathcal{D}$ be region between the lines $y=-x$, $y=-1$ and $x=-1$. Compute the integral

$$\iint_{\mathcal{D}} 2y \ \mathrm{d}A.$$

Solution:

$$
\begin{aligned}
\iint_{\mathcal{D}} 2y \; \mathrm{d}A &= \int_{-1}^{1} \int_{-1}^{-x} 2y \; \mathrm{d}y \; \mathrm{d}x, \\
&= \int_{-1}^{1} y^2 \Big|_{-1}^{-x} \; \mathrm{d}x, \\
&= \int_{-1}^{1} \int_{-1}^{-x} x^2 - 1 \; \mathrm{d}x, \\
&= x^3/3 - x \Big|_{-1}^{1}, \\
&= -4/3
\end{aligned}
$$