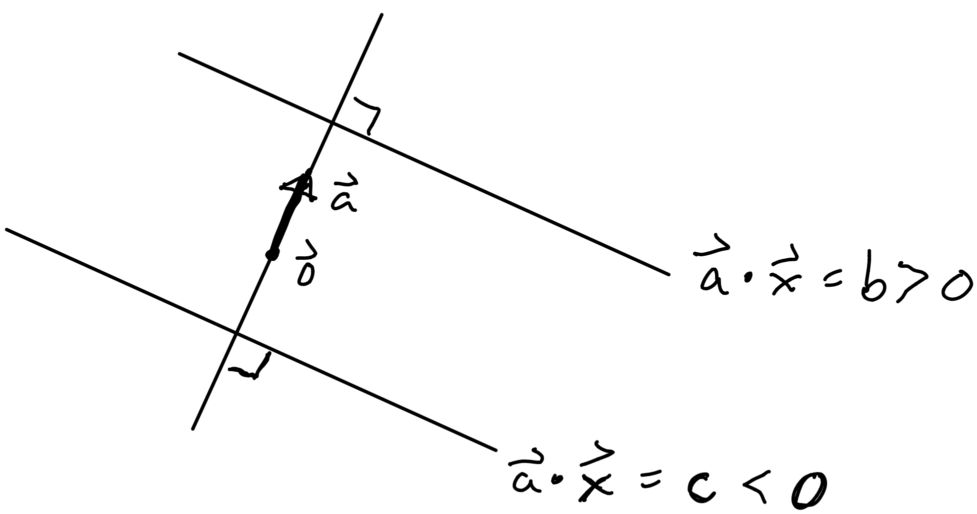
Today: HW2 Discussion
Review for Quiz 2

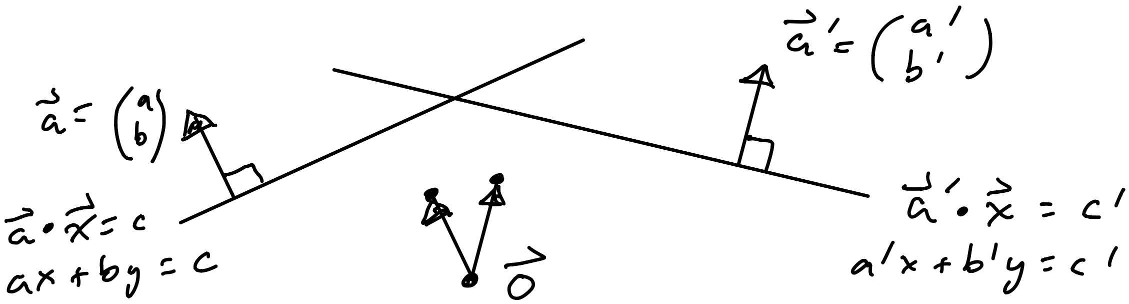
[Quiz 2: First 25 minutes of Monday's class.]

HW2 Discussion:

Problem 1:



Problem 2:



(a) lines are $\perp$

$\iff$ vectors $\vec{a}$ & $\vec{a}'$ are $\perp$

$\iff$ $\vec{a} \cdot \vec{a}' = 0$

$\iff$ $aa' + bb' = 0.$ ///

(b) Lines are $\parallel$

$\iff$ vectors $\vec{a}$ & $\vec{a}'$ are $\parallel$

$\iff$ $\vec{a}' = t\vec{a}$ for some t.

$(a', b') = (ta, tb)$

$$\begin{cases} a' = ta \\ b' = tb. \end{cases}$$

$\iff$ $\dfrac{a}{a'} = \dfrac{b}{b'}$ or $a = a' = 0$

or $b = b' = 0$

$\iff$ $\boxed{ab' = a'b}$

This single equation includes all 3 cases!

Let me introduce another notation.

Given constants $a, b, c, d \in \mathbb{R}$ we define the determinant of a 2×2 matrix as follows:

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} := ad - bc$$

This symbol means "is defined to be..."

Using this language we have

vectors $\vec{a} = \begin{pmatrix} a \\ b \end{pmatrix}$ & $\vec{a}' = \begin{pmatrix} a' \\ b' \end{pmatrix}$ are parallel

$$\iff ab' = a'b$$

$$\iff ab' - a'b = 0$$

$$\iff \det \begin{pmatrix} a & b \\ a' & b' \end{pmatrix} = 0$$

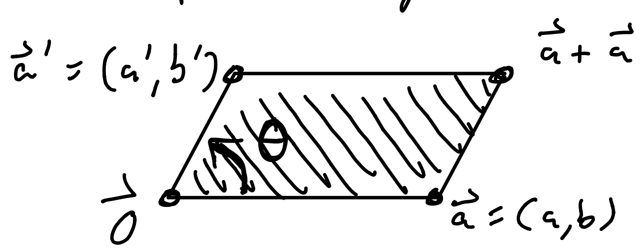
$$\iff \det \begin{pmatrix} a & a' \\ b & b' \end{pmatrix} = 0.$$

[Remark: A determinant is a sort of magic trick to detect when vectors are parallel.]

In fact, there is a good geometric reason for this.

<u>Theorem</u>:

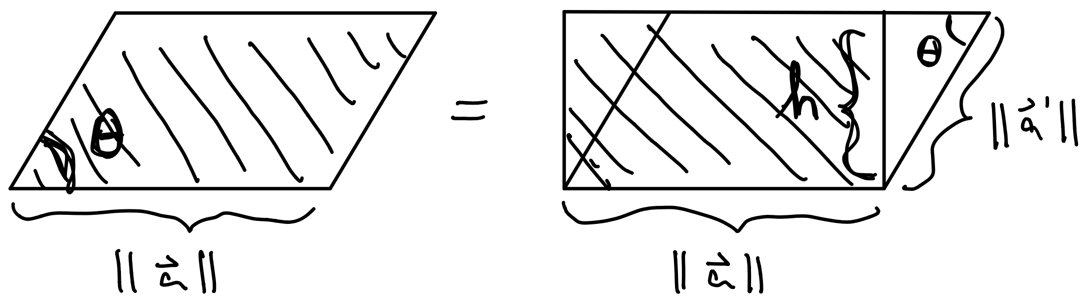
$$\det\begin{pmatrix} a & a' \\ b & b' \end{pmatrix} \overset{(*)}{=} \pm \text{ the } \underline{\text{area}} \text{ of the parallelogram}$$



$$\overset{(**)}{=} \|\vec{a}\| \, \|\vec{a}'\| \sin\theta.$$

[Direction of θ (clockwise or counter-clockwise) accounts for the $\pm$.]

<u>Proof</u>: (**)



height h satisfies :

$$\sin\theta = \frac{h}{\|\vec{a}'\|}$$

$$h = \|\vec{a}'\| \sin\theta$$

Therefore the area is

base $\cdot$ height $= \|\vec{a}\| \cdot \|\vec{a}'\| \sin\theta$. ✓

Maybe we'll prove (*) later. ///

We observe again that

$\vec{a} = (a,b)$ & $\vec{a}' = (a',b')$ are parallel

$\Longleftrightarrow \quad \theta = 0°$ or $180°$

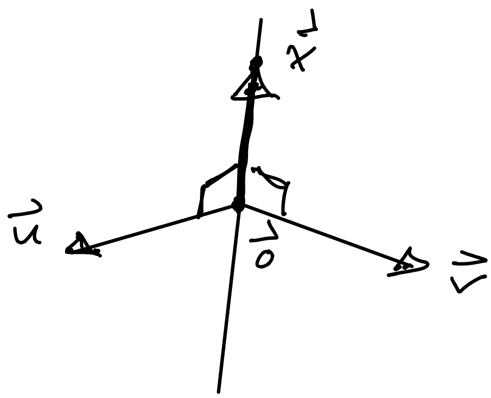
$\Longleftrightarrow \sin\theta = 0 \Longleftrightarrow \det\begin{pmatrix} a & a' \\ b & b' \end{pmatrix} = 0.$

Problem 4: The Cross Product.

Given vectors $\vec{u}$ & $\vec{v}$ in 3D, the linear system

$$\begin{cases} \vec{u} \cdot \vec{x} = 0 \\ \vec{v} \cdot \vec{x} = 0 \end{cases}$$

represents a <u>line</u> (assume $\vec{u}, \vec{v}$ not parallel)



We would like to choose one special vector on this line.

The most natural choice is

called the <u>cross product</u>:

$$\vec{u} \times \vec{v} := \left(u_2 v_3 - u_3 v_2 ,\ u_3 v_1 - u_1 v_3 ,\ u_1 v_2 - u_2 v_1 \right)$$

[Remark:

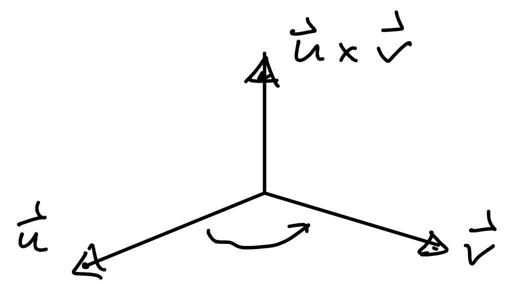
(vector in $\mathbb{R}^3$) $\times$ (vector in $\mathbb{R}^3$)

= (vector in $\mathbb{R}^3$) .]

Why is this a "natural" choice ?

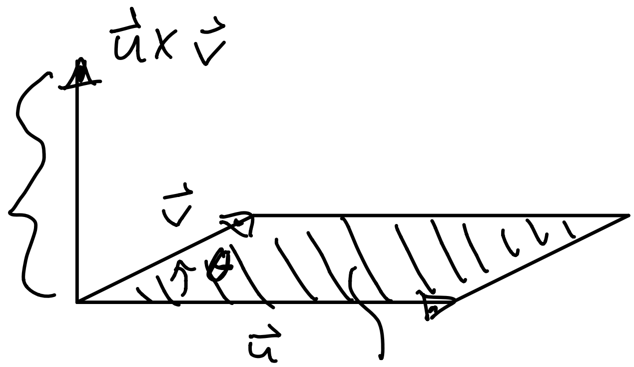
Two reasons:

- Direction: Right Hand Rule



- Length: Area of Parallelogram generated by $\vec{u}$ & $\vec{v}$:

length
$\|\vec{u}\|\|\vec{v}\|\sin\theta$



area
$\|\vec{u}\|\|\vec{v}\|\sin\theta$.

Consequence:

$\vec{u}$ & $\vec{v}$ are parallel

$$\iff \sin\theta = 0$$

$$\iff \|\vec{u}\times\vec{v}\| = 0$$

$$\iff \vec{u}\times\vec{v} = (0,0,0)$$

$$\iff \begin{cases} u_2v_3 = u_3v_2 \\ u_1v_3 = u_3v_1 \\ u_1v_2 = u_2v_1 \end{cases}$$

Compare to previous discussion of vectors in 2D.

Guess : Vectors $\vec{a} = (a_1, a_2, \dots, a_n)$ & $\vec{a}' = (a_1', a_2', \dots, a_n')$ are <u>parallel</u>

$$\Updownarrow$$

$$a_i a_j' = a_i' a_j \quad \text{for all } i \neq j .$$

Comare with the Known Fact:

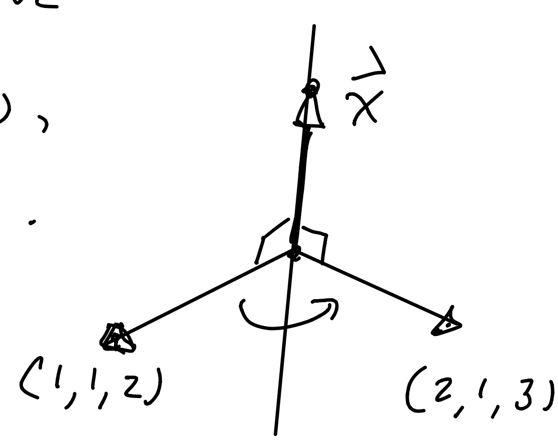
$\vec{a}$ & $\vec{a}'$ are pependicular

$$\iff \vec{a} \cdot \vec{a}' = 0$$

$$\iff a_1 a_1' + a_2 a_2' + \cdots + a_n a_n' = 0 .$$

Problem 4(b) : Solve

$$\begin{cases} x + y + 2z = 0, \\ 2x + y + 3z = 0 . \end{cases}$$



Solution is $\vec{x} = t\,\vec{a}$

$$(x, y, z) = t\,(a, b, c)$$

where $\vec{a}$ is the cross product:

$$\vec{a} = (1, 1, 2) \times (2, 1, 3)$$

Mnemonic Device:

$$\vec{u} \times \vec{v} \text{ "="} \det \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix}$$

$$= \hat{i} \det \begin{pmatrix} u_2 & u_3 \\ v_2 & v_3 \end{pmatrix} - \hat{j} \det \begin{pmatrix} u_1 & u_3 \\ v_1 & v_3 \end{pmatrix} + \hat{k} \det \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix}$$

where $\hat{i}, \hat{j}, \hat{k}$ are physics notation for the standard basis vectors:

$$\hat{i} = (1, 0, 0)$$
$$\hat{j} = (0, 1, 0)$$
$$\hat{k} = (0, 0, 1).$$

In our case:

$$\det\begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{pmatrix} = (3-2,\, 4-3,\, 1-2)$$
$$= (1, 1, -1)$$

So the solution is

$$(x, y, z) = t(1, 1, -1).$$

This is a parametrized <u>line</u>.

Problem 5: Solve

$$\begin{cases} ① \; x + y + 2z = 0, \\ ② \; 2x + y + 3z = 0, \\ ③ \; 2x + 3y + cz = 4. \end{cases}$$

Already know:

$$\left.\begin{cases} ① \; x + y + 2z = 0 \\ ② \; 2x + y + 3z = 0 \end{cases}\right\} \iff \begin{pmatrix} x \\ y \\ z \end{pmatrix} = t\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}.$$

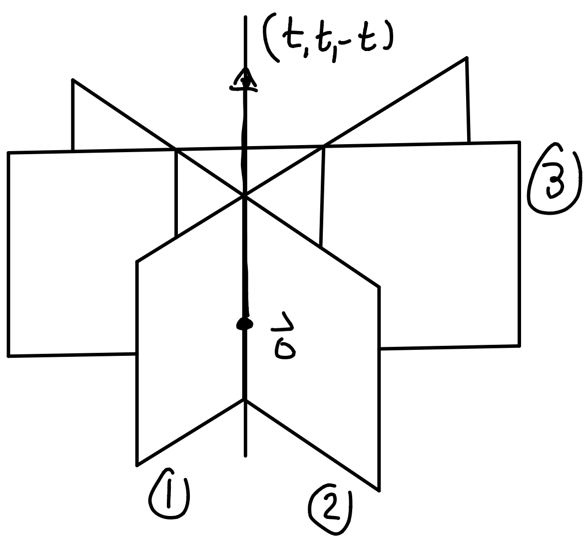
We need the intersection of this line with the plane ③:

$$2x + 3y + cz = 4$$
$$2(t) + 3(t) + c(-t) = 4$$
$$(5-c)t = 4.$$

Two cases:

- If $c = 5$ then this equation has NO SOLUTION.

  Meaning: The line $(x,y,z) = t(1,1,-1)$ is parallel to (and not contained in) the plane 3. However, in this case, no two of the planes 1, 2, 3 are parallel. Picture:



Plane 3 is parallel to the line of intersection of 1 & 2.

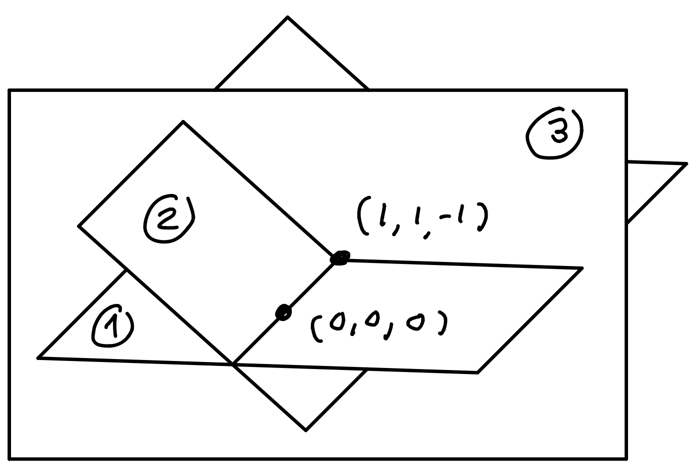
- If $c \neq 5$ then $t = 4/(5-c)$ and there is a <u>unique solution</u>

$$(x,y,z) = \frac{4}{5-c}(1,1,-1).$$

In particular, if $c=1$ then

$$(x,y,z) = (1,1,-1).$$

Picture:



Note that plane 3 does not contain the origin.

For Quiz: Systems of linear equations in 2 or 3 variables.
Understand how this is related to geometry of lines & planes in 2D and 3D.
The concepts of dot product and cross product are relevant!