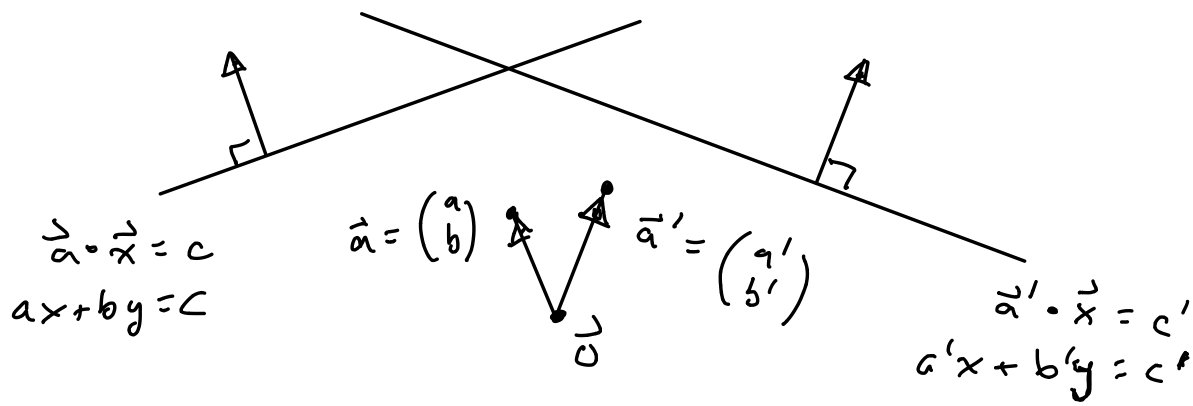
HW2 due <u>tomorrow</u> before class.

Hints for Problem 2:



We observe that

lines are perpendicular

$\iff$ vectors $\vec{a}$ & $\vec{a}'$ are perpendicular.

$\iff$ $\vec{a} \cdot \vec{a}' = 0$

$aa' + bb' = 0$.

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lines are parallel

$\iff$ vectors $\vec{a}$ & $\vec{a}'$ point in the same direction (or opposite directions)

$\Longleftrightarrow$ $\underbrace{\vec{a}' = t\vec{a} \text{ for some nonzero scalar } t}$.

Try to eliminate t to obtain a single equation involving a, b, a', b'.

Last time we discussed systems of 2 or 3 linear equations in 3 unknowns.

Example:
$$\begin{array}{l} ① \\ ② \end{array} \begin{cases} x + y + z = 1 \\ 0 + y + 2z = 3 \end{cases}$$

$$\rightsquigarrow \quad \begin{array}{l} ①' = ① - 1② \\ ②' = ② \end{array} \begin{cases} x + 0 - z = -2 \\ 0 + y + 2z = 3 \end{cases}$$

Done ✓

pivot variables   free variable

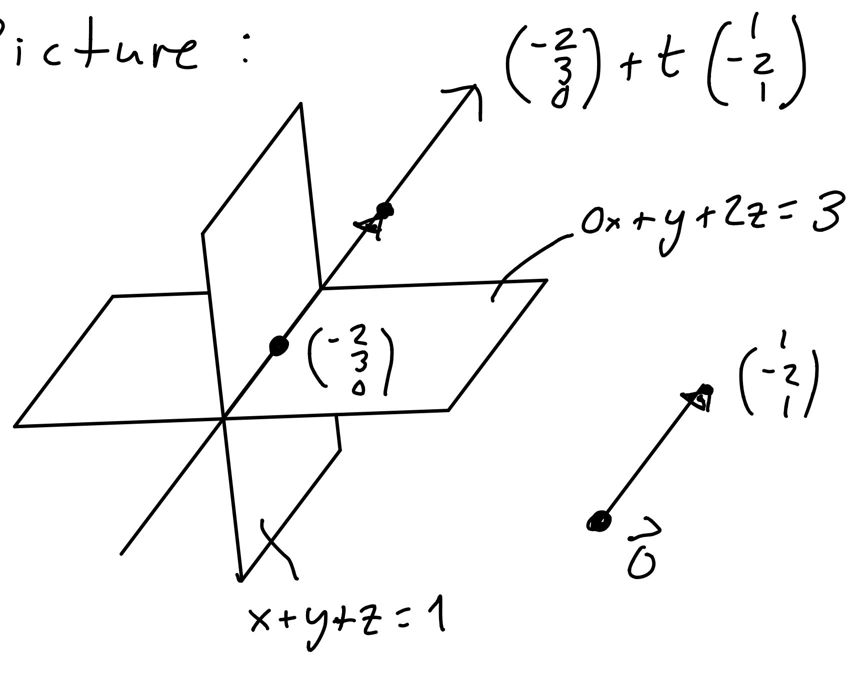
Solve for x, y, z in terms of the free variable z:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2+z \\ 3-2z \\ z \end{pmatrix}$$

$$= \begin{pmatrix} -2+1z \\ 3-2z \\ 0+1z \end{pmatrix}$$

$$= \begin{pmatrix} -2 \\ 3 \\ 0 \end{pmatrix} + z \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \quad \text{a line !}$$

Picture :



Conventional to rename the free variables as t or s or $t_1, t_2, t_3, \ldots$

[Remark: t is for "time".]

I have said that 1 linear equation in 3 unknowns represents a "2-dimensional plane."

But what does this mean?

Consider the equation

$$\left\{ x + 2y + 3z = 1 \right.$$

pivot variable

free variables.

Solve for x, y, z in terms of the free variables y, z:

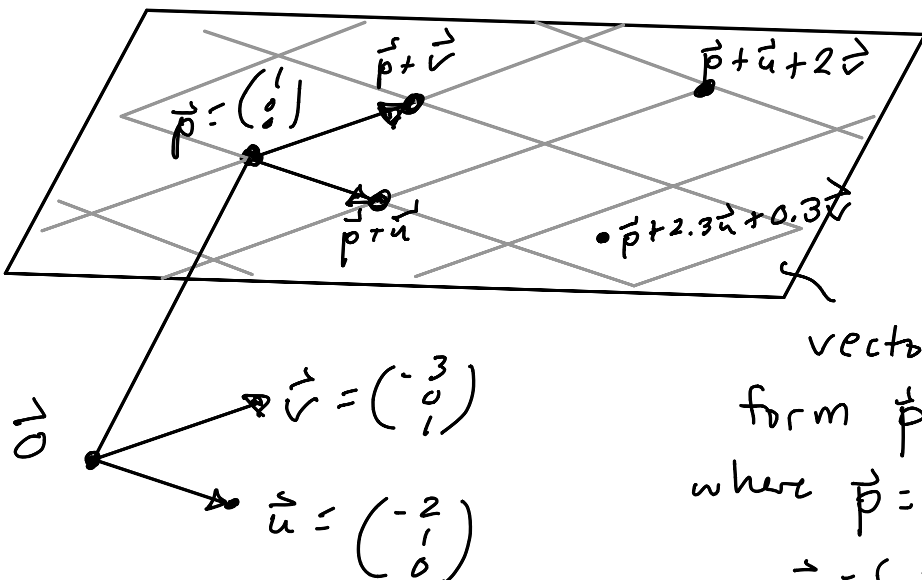
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 - 2y - 3z \\ y \\ z \end{pmatrix}$$

$$= \begin{pmatrix} 1 - 2y - 3z \\ 0 + 1y + 0z \\ 0 + 0y + 1z \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix}1\\0\\0\end{pmatrix} + y\begin{pmatrix}-2\\1\\0\end{pmatrix} + z\begin{pmatrix}-3\\0\\1\end{pmatrix}}_{}$$

What is this?

A "parametrized plane"



vectors of the form $\vec{p} + s\vec{u} + t\vec{v}$ where $\vec{p} = (1,0,0)$

$\vec{u} = (-2,1,0)$

$\vec{v} = (-3,0,1)$.

This explains why we say a plane is "2-dimensional": it has a coordinate system consisting of 2 "basis vectors" $\vec{u}$ & $\vec{v}$.

Remarks:

- The coordinate system is not unique. We could use any two vectors in the plane, as long as they are not parallel.

- Every point on the plane has a unique address in terms of the basis vectors.

- There is no "standard basis" for this plane. Since there is no best choice, we are forced to make an arbitrary choice, such as

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}.$$

Important General Definition:

Let $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_d \in \mathbb{R}^n$. We say these vectors are (linearly) <u>independent</u> if they have no (linear) <u>relations</u>.

That is, for all scalars $t_1, t_2, \ldots, t_d$ we have

$$t_1\vec{u}_1 + t_2\vec{u}_2 + \cdots + t_d\vec{u}_d = \vec{0}$$

$$\implies t_1 = t_2 = \cdots = t_d = 0.$$

[Example: The vectors $\vec{u}_1$ & $\vec{u}_2$ are independent if and only if

$$\vec{u}_1 \neq t\vec{u}_2 \text{ for any } t,$$

i.e. when they are <u>not parallel</u>.]

If $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_d$ are in<u>depen</u>dent then for any point $\vec{p}$ we say that

$$\{\vec{p} + t_1\vec{u}_1 + t_2\vec{u}_2 + \cdots + t_d\vec{u}_d \,;\, t_1, \ldots, t_d \in \mathbb{R}\}$$

is a "d-dimensional plane" (or just a "d-plane").

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This is the official mathematical definition of the word

"dimension."

This leads to the very important

Dimension Principle

The solutions to a system of m linear equations in n unknowns always form a d-plane for some $d \in \{0, 1, 2, \ldots, n\}$, or there is no solution. Moreover:

- If $m \leq n$ then we <u>probably</u> have

$$d = n - m$$

dimension of the solution set = # variables − # equations.

[Example: $m = 2$ planes in $n = 3$ dimensional space probably meet in a $(n - m) = 1$ dimensional line.]

- If $m > n$ (# equations > # variables) then there is <u>probably</u> NO SOLUTION.

These general ideas allow us to go beyond 2 & 3 dimensional space.

Example:

$$\begin{cases} (1) & \underbrace{x_1 + 0}_{\text{pivot variables}} + \underbrace{x_3 + x_4}_{\text{free variables}} = 1 \\ (2) & 0 + x_2 + 2x_3 + 3x_4 = 5 \end{cases}$$

Solve for x_1, x_2, x_3, x_4 in terms of the free variables x_3, x_4:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 - x_3 - x_4 \\ 5 + 2x_3 - 3x_4 \\ x_3 \\ x_4 \end{pmatrix}$$

$$= \begin{pmatrix} 1 - 1x_3 - 1x_4 \\ 5 + 2x_3 - 3x_4 \\ 0 + 1x_3 + 0x_4 \\ 0 + 0x_3 + 1x_4 \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} 1 \\ 5 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 2 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ -3 \\ 0 \\ 1 \end{pmatrix}}_{\text{a ``2-dimensional plane'' living in ``4-dimensional space.''}}$$

We could draw a picture, but it wouldn't be very accurate.

Next Step: Solve a more general looking system.

$$\begin{cases} ① & x_1 + 3x_2 + 0 + 2x_4 = 1 \\ ② & x_1 + 3x_2 + x_3 + 6x_4 = 7 \end{cases}$$

It is not clear which variables are "pivot" and which are "free."

So we perform "Gaussian Elimination":

Details: Look in the top left.

we see x_1, so that is our first pivot.

Now apply "row operations" to eliminate below the pivot:

$$\begin{aligned} ①' = ① \quad & \begin{cases} x_1 + 3x_2 + 0 + 2x_4 = 1 \\ 0 + 0 + x_3 + 4x_4 = 6 \end{cases} \quad \begin{array}{|l} x_1 = 1 - 3x_2 - 2x_4 \\ x_3 = 6 - 4x_4 \end{array} \\ ②' = ② - 1① \quad & \end{aligned}$$

oops

Next look for the top left nonzero entry below the 1st pivot.

This is our second pivot.

Since there are no more equations, we are done. ✓

Conclusion:

Pivot variables: x_1 & x_3

Free variables: x_2 & x_4.

Solution:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 - 3x_2 - 2x_4 \\ x_2 \\ 6 - 4x_4 \\ x_4 \end{pmatrix}$$

$$= \begin{pmatrix} 1 - 3x_2 - 2x_4 \\ 0 + 1x_2 + 0x_4 \\ 6 + 0x_2 - 4x_4 \\ 0 + 0x_2 + 1x_4 \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} 1 \\ 0 \\ 6 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ 0 \\ -4 \\ 1 \end{pmatrix}}_{\text{A parametrized 2-plane living in } \mathbb{R}^4.}$$

In general, we can perform Gaussian Elimination to put any linear system into so-called "Reduced Row-Echelon Form" (RREF)

$$
\left\{
\begin{array}{ccccccccccccccc}
 & & x_2 & & x_3 & & & & x_5 & & & & x_7 & & \\
x_1 & + & ? & + & ? & + & 0 & + & ? & + & 0 & + & ? & = & ? \\
0 & + & 0 & + & 0 & + & x_4 & + & ? & + & 0 & + & ? & = & ? \\
0 & + & 0 & + & 0 & + & 0 & + & 0 & + & x_6 & + & ? & = & ? \\
0 & + & 0 & + & 0 & + & 0 & + & 0 & + & 0 & + & 0 & = & c
\end{array}
\right.
\qquad \text{some const.}
$$

If $c \neq 0$ then the final equation is FALSE, hence the system has no solution.

If $c = 0$, then there *is* a solution of the form

$$\vec{p} + x_2 \vec{u} + x_3 \vec{v} + x_5 \vec{w} + x_7 \vec{r},$$

where the free parameters/variables are x_2, x_3, x_5, x_7.

If you like, the solution forms a "4-dimensional parametrized plane" living in "7-dimensional space."