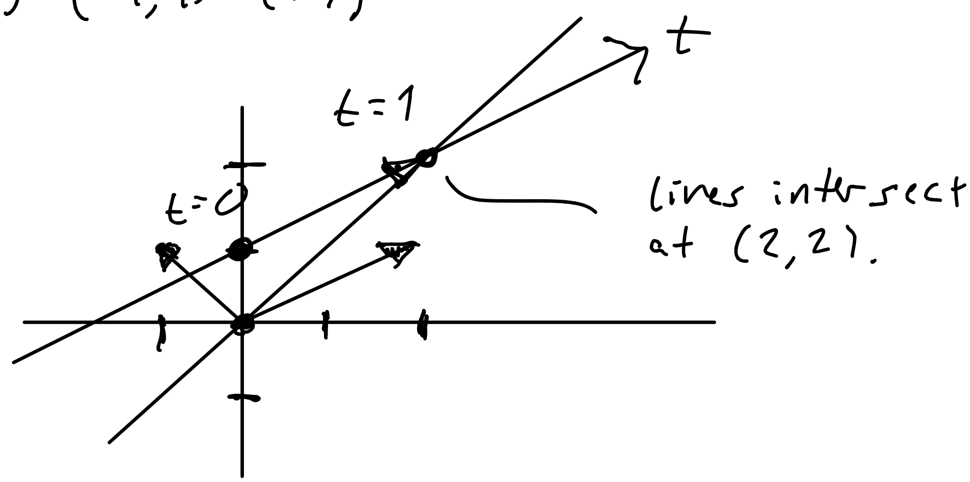
HW 2 due Friday before class.

Quiz 1 Discussion:

Problem 2: Drawing Lines

(a) $(x, y) = (0, 1) + t(2, 1)$

(b) $(-1, 1) \cdot (x, y) = 0$



$$(-1, 1) \cdot (x, y) = 0$$
$$-x + y = 0$$
$$x = y.$$

Alternatively, we can calculate the point of intersection:

$$(x, y) = (0, 1) + t(2, 1) = (2t, 1+t).$$

Plug into $-x + y = 0$

$$-(2t) + (1+t) = 0$$
$$1 - t = 0$$

$t = 1$.

Therefore the point of intersection is $(x, y) = (2t, 1+t) \underset{(t=1)}{=} (2, 2)$ ✓

This Week & Next:

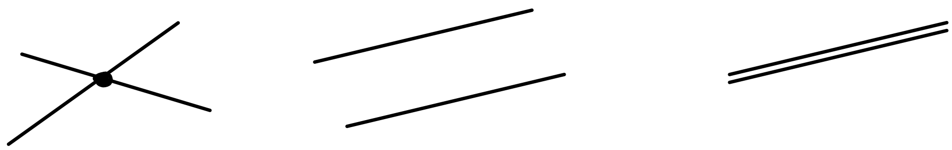
Solving a system of m simultaneous linear equations in n unknowns.

Geometrically, we can (but we don't have to) think of this as the intersection of m $(n-1)$ dimensional hyperplanes living in n-dimensional Cartesian space $\mathbb{R}^n$.

This picture helps our understanding when m & n are small.

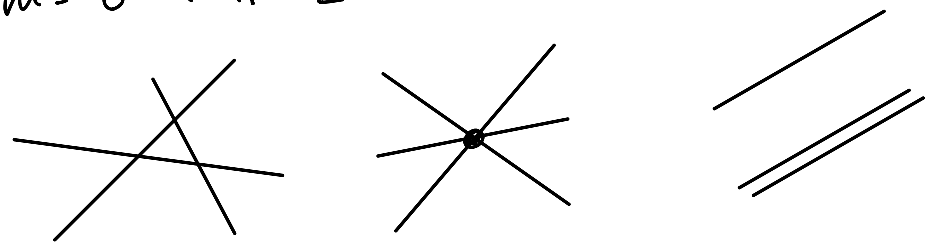
Recall:

- $m = 2$ & $n = 2$



Most likely a unique solution

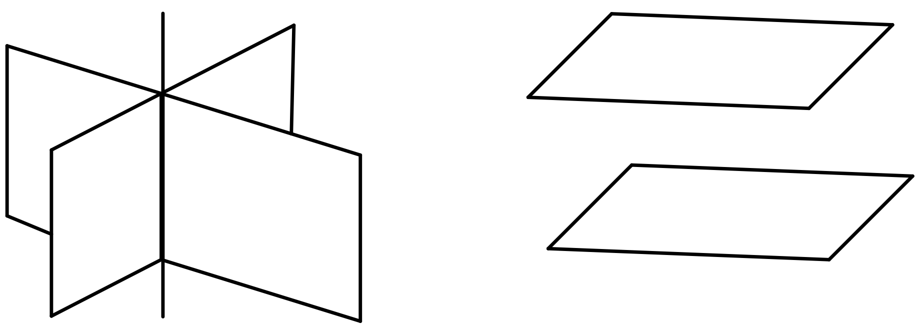
- $m = 3$ & $n = 2$



etc.

Most likely NO SOLUTION.

- $m = 2$ & $n = 3$



Most likely a line of solutions.

Example: $\begin{cases} x + y + z = 0, \\ x + 2y + 3z = 0. \end{cases}$

The algorithm that we use to solve this is called "Gaussian Elimination."

[Gauss invented this method to compute "least squares regression." He used it to predict the orbit of asteroid Ceres.]

Main Idea: Given a linear system, if we replace i^{th} equation by

$$(i^{th} \text{ equation}) - \underset{\substack{\uparrow \\ \text{some constant}}}{\lambda}\,(j^{th} \text{ equation}),$$

then the solutions of the system <u>stay the same</u>. By repeated applications we can obtain an <u>equivalent</u> system that is much <u>simpler</u>.

In our example:

$$\begin{array}{l} ① \\ ② \end{array}\begin{cases} x+y+z=0 \\ x+2y+3z=0 \end{cases} \rightsquigarrow \begin{array}{r} ① \\ ③=②-1① \end{array}\begin{cases} x+y+z=0 \\ 0+y+2z=0 \end{cases}$$

$$\rightsquigarrow \begin{array}{r} ④=①-1③ \\ ③ \end{array}\begin{cases} x+0-z=0 \\ 0+y+2z=0 \end{cases}.$$

Remarks:

- We cannot simplify any further.
- The simpler system (3) & (4) has the same solutions as the original system (1) & (2).
- The solution of the simpler system is easy to read off:

x & y are called "pivot variables" and z is called a "free variable" (or a "parameter").

<u>Solve</u> for the pivot variables in terms of the free variables.

$$\left.\begin{aligned} x &= z \\ y &= -2z \\ (z &= z) \end{aligned}\right\}$$

What kind of shape does this represent?

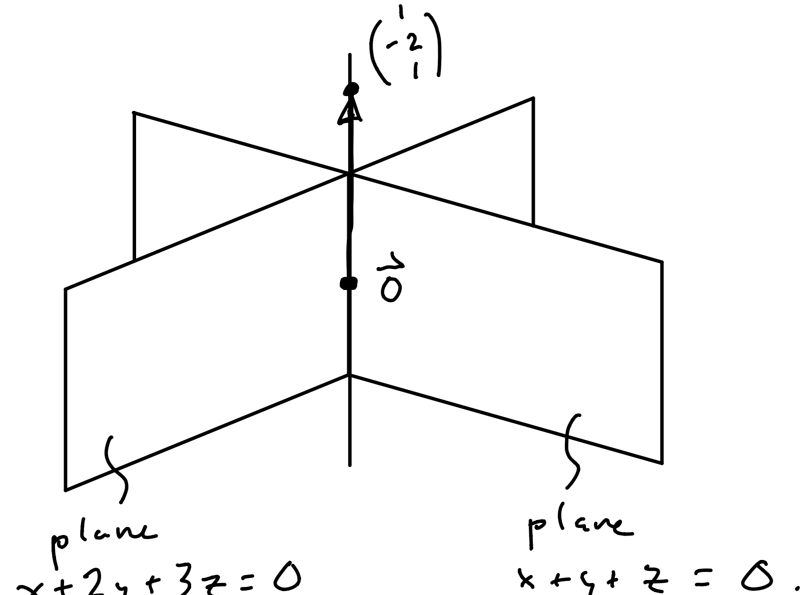
This is a "parametrized line":

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ -2z \\ z \end{pmatrix} = z \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$

The solution is "parametrized" by z. It is common to rename the parameters, say $t = z$. Then the solution is

$$(x, y, z) = t(1, -2, 1).$$

Picture:



Different way to Think:

$$\begin{cases}(1,1,1)\cdot(x,y,z)=0\\(1,2,3)\cdot(x,y,z)=0\end{cases}$$

means (x,y,z) is simultaneously $\perp$ to $(1,1,1)$ & $(1,2,3)$.

There is a general formula for this.

Given vectors $\vec{u} = (u_1, u_2, u_3)$ &

$\vec{v} = (v_1, v_2, v_3)$ we define the <u>cross product</u>

vector:

$$\vec{u} \times \vec{v} = \left(u_2 v_3 - u_3 v_2 , \underbrace{u_3 v_1 - u_1 v_3}_{\substack{\text{second coordinate} \\ \text{looks backwards,} \\ \text{but it's not.}}} , u_1 v_2 - u_2 v_1 \right)$$

You will verify on HW2 that

$$\vec{u} \bullet (\vec{u} \times \vec{v}) = 0$$

$$\vec{v} \bullet (\vec{u} \times \vec{v}) = 0,$$

i.e. vector $\vec{u} \times \vec{v}$ is simultaneously

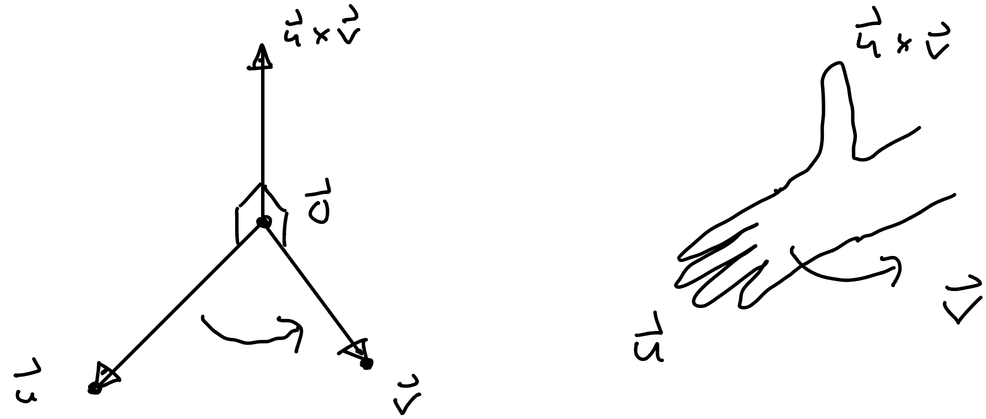
$\perp$ to $\vec{u}$ & $\vec{v}$.

Example: $\vec{u} = (1, 1, 1)$
$\vec{v} = (1, 2, 3)$.

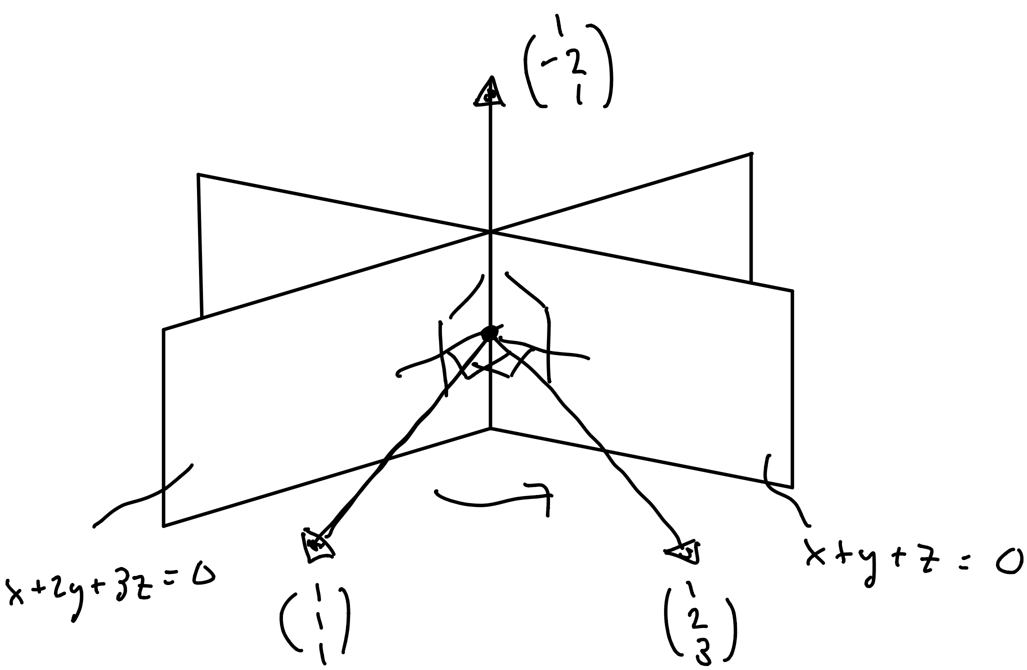
$$\vec{u} \times \vec{v} = \left(\underset{3-2}{u_2 v_3 - u_3 v_2} , \underset{1-3}{u_3 v_1 - u_1 v_3} , \underset{2-1}{u_1 v_2 - u_2 v_1} \right)$$

$$= (1, -2, 1)$$

Where have we seen this before?



Right Hand Rule.



Any vector that is $\perp$ to $(1,1,1)$ & $(1,2,3)$ is on <u>both</u> of the planes $(1,1,1)\cdot(x,y,z)=0$ & $(1,2,3)\cdot(x,y,z)=0$.

The cross product gives quick way to compute the solution when

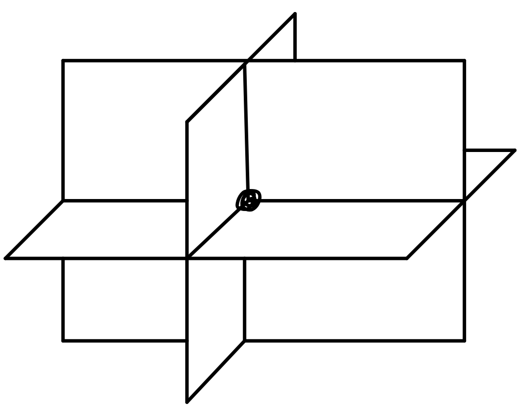
$m = \#$ equations $= 2$

$n = \#$ unknowns $= 3$.

Next: $m = 3$ & $n = 3$

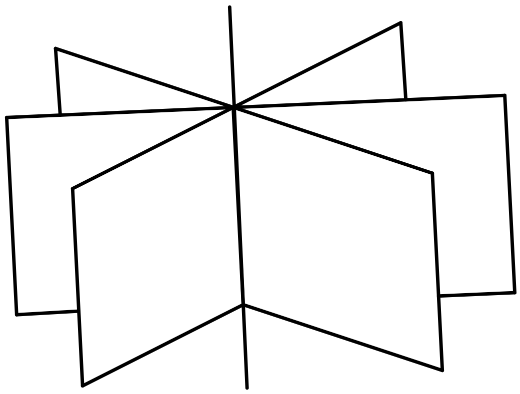
Intersection of 3 planes in 3D.

What are the possibilities?

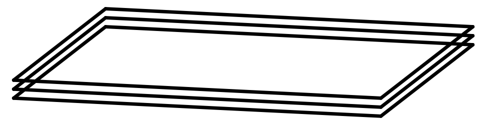


Most likely a unique solution, i.e., a single point of intersection.

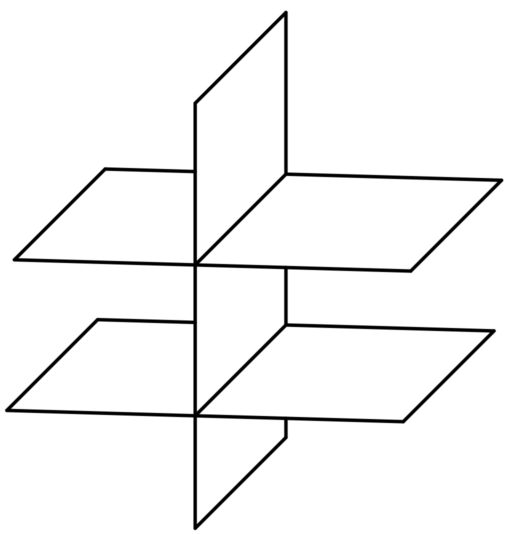
But we could also have a line of solutions or a plane of solutions or no solutions:



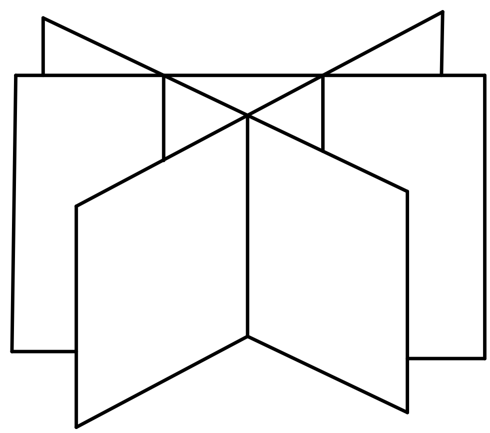
∞ many solutions forming a line



∞ many solutions forming a plane



2 planes parallel means no solution



Even though no 2 planes are parallel, we still have no solution.

As you can imagine, higher dimensions get even more complicated, but we cannot draw the pictures. Nevertheless, we can solve large systems of linear equations using

ALGEBRA.

Add a 3rd plane to our example:

$$\begin{array}{ll} (1) & x+y+z=0 \\ (2) & x+2y+3z=0 \\ (3) & x+2y+cz=1 \end{array}$$

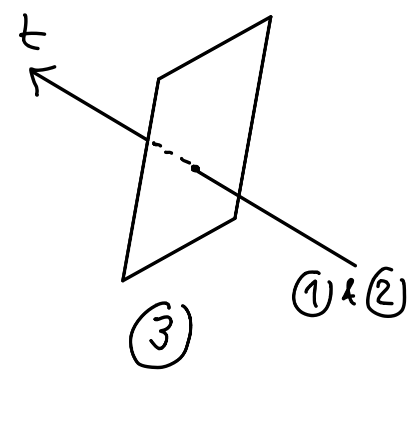
$\boxed{c \text{ is a constant.}}$

Solve for x, y, z.

We already know (1) & (2) together is equivalent to a <u>line</u>

$$(x,y,z) = t(1,-2,1) = (t,-2t,t).$$

Plug into (3) to get



$$x + 2y + cz = 1$$
$$(t) + 2(-2t) + c(t) = 1$$
$$-3t + ct = 1$$
$$(c-3)t = 1.$$

What does this mean? Two Cases:

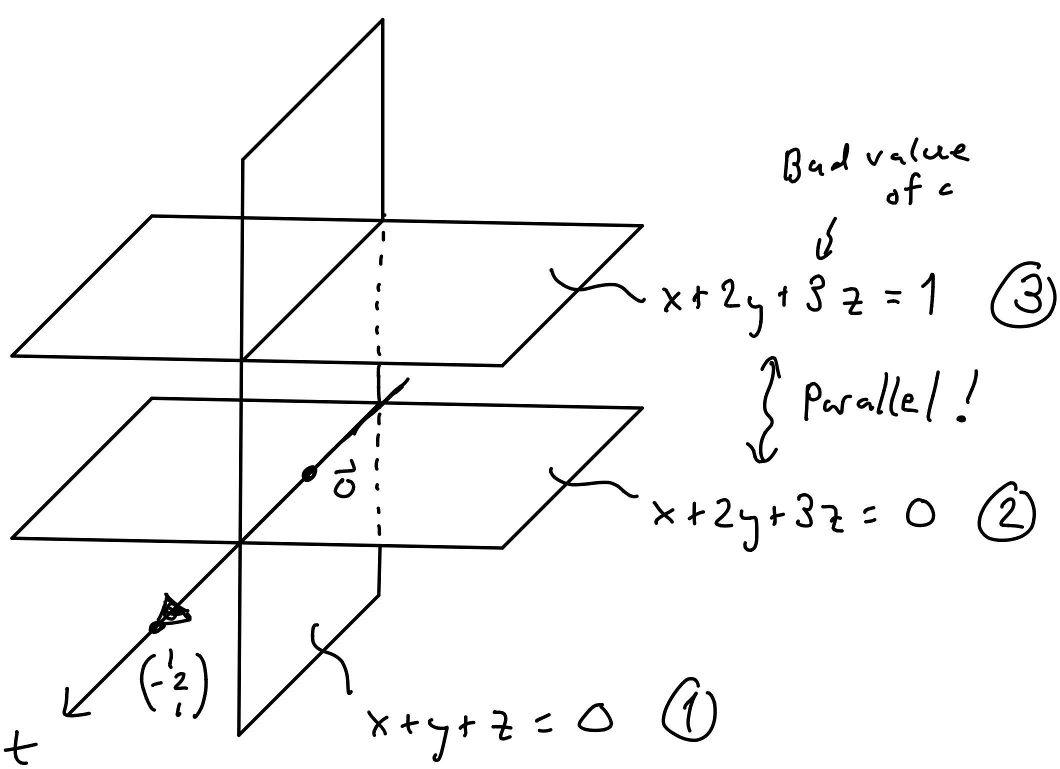
- If $c \neq 3$ then $t = 1/(c-3)$ and the solution is a unique point

$$(x,y,z) = t(1,-2,1) = \frac{1}{c-3}(1,-2,1).$$

- If $c = 3$ then the equation

$$(c-3)t = 1$$

has <u>no solution</u>, hence the original system has <u>no solution</u>.



The line of intersection of planes 1 & 2 is parallel (hence never intersects) with plane 3.

This was caused because planes 2 & 3 are parallel.