HW 2 on the webpage,
due <u>before</u> class on Friday May 29.

This week: Solving systems of Linear equations.

Recall: A <u>linear</u> equation in n unknowns $x_1, x_2, \ldots, x_n$ has the form

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b$$

for some constants $a_1, a_2, \ldots, a_n$ & b.

We can also write this as

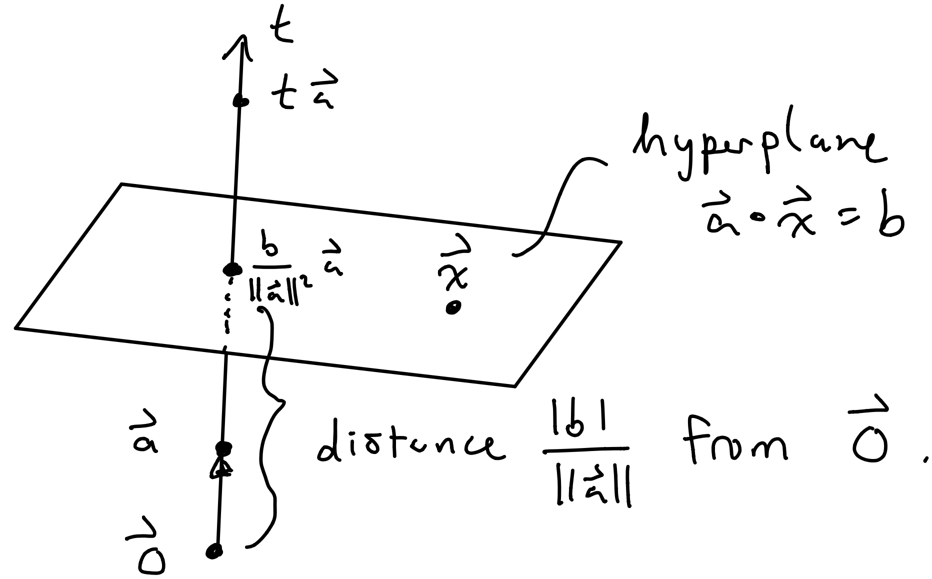
$$\vec{a} \cdot \vec{x} = b$$

where $\vec{a} = (a_1, a_2, \ldots, a_n)$ is a constant vector and

$$\vec{x} = (x_1, x_2, \ldots, x_n)$$

is a vector of unknowns.

Geometrically, this equation represents an "$(n-1)$-dimensional hyperplane" living in n-dimensional space:



The Main Problem of Linear Algebra is to solve a system of m simultaneous linear equations in n unknowns:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \qquad\qquad\qquad\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

Solve for $x_1, x_2, \ldots, x_n$!

What does this mean?

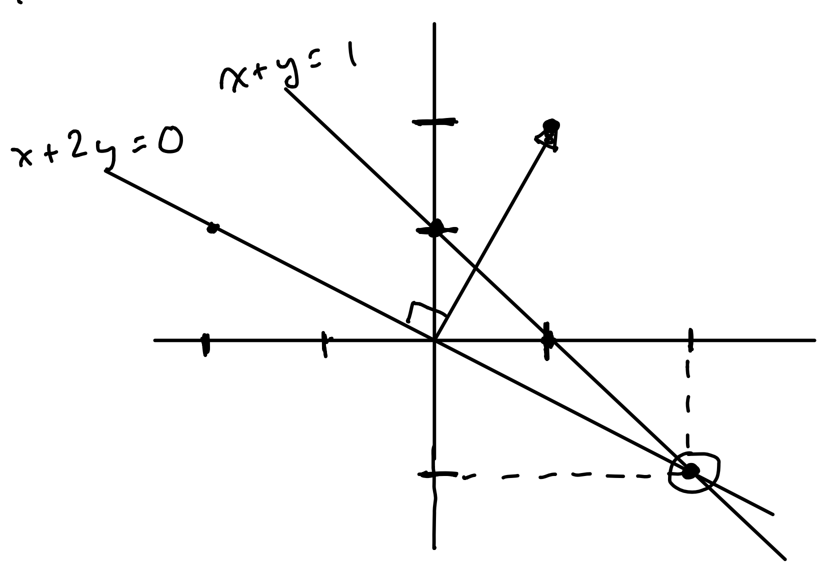
For small values of m & n it is possible to draw pictures.

Example: $m = 2$ & $n = 2$.

$$\begin{cases} x + y = 1, \\ x + 2y = 0. \end{cases}$$

Geometrically, this system repesents the intersection of two lines in the Cartesian plane.

Picture:



Compute:

$$\begin{cases} 1x + y = 1 & (1) \\ 1x + 2y = 0 & (2) \end{cases}.$$

Method of elimination: Find a new equation with <u>no</u> x term.

Consider equation

$$(3) = (2) - (1)$$

$$\begin{array}{rrcrcl} & (2) & x & + & 2y & = 0 \\ - & (1) & x & + & y & = 1 \\ \hline & (3) & 0 & + & y & = -1 \end{array}$$

Back-substitute $y = -1$ into either (1) or (2). Into (1):

$$x + y = 1$$
$$x + (-1) = 1$$
$$x = 2 \quad \checkmark$$

The lines / linear equations (1) & (2) intersect at the single point

$$(x, y) = (2, -1) \quad \checkmark$$

Let's change it a bit:

$$\begin{array}{l} (1) \\ (2) \end{array} \begin{cases} x + y = 1 \\ x + cy = 0 \end{cases}$$

where c is some constant.

Elimination:

(3) = (2) − (1) :

$$0x + (c-1)y = -1$$

Now there are two cases:

- If $c - 1 \neq 0$ then we get a unique solution $y = 1/(1-c)$

$$x = 1 - y = \frac{1-c}{1-c} - \frac{1}{1-c}$$

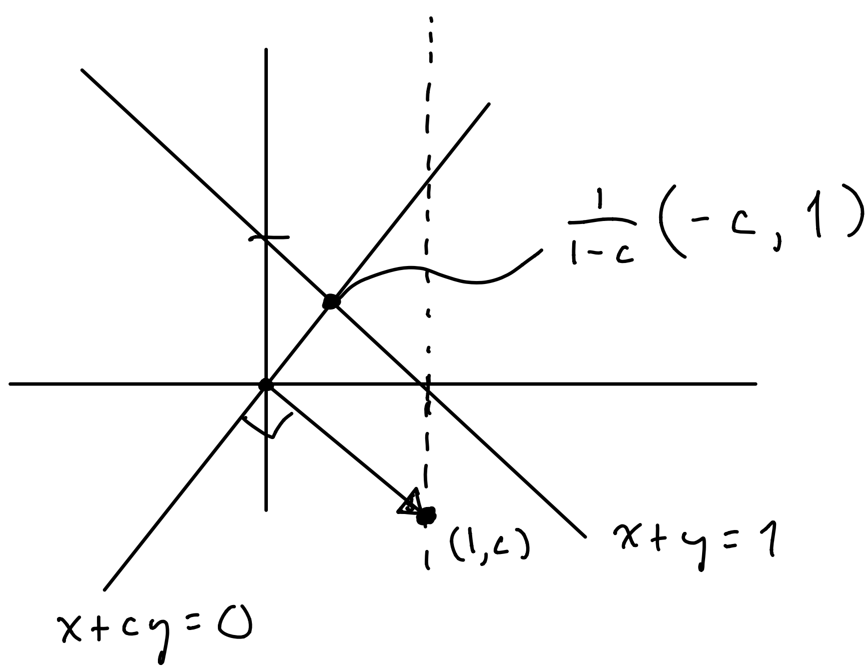
$= -c/(1-c)$.

- If $c-1=0$, then the "true" equation (3) says that

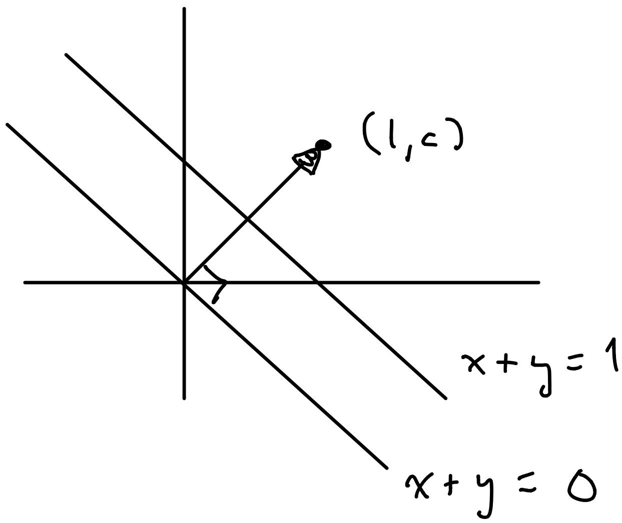
$$0x + 0y = -1$$

This is impossible, which means that the original system (1) & (2) has NO SOLUTION!

Picture: $c \neq 1$:

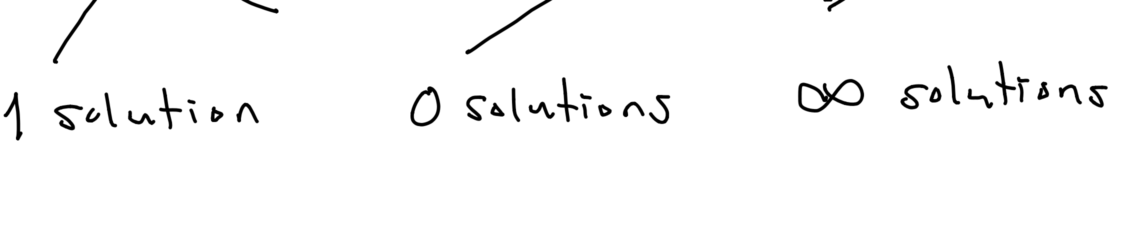


$c = 1$:



In this case the lines are <u>parallel</u>, so they do not intersect, i.e., the equations (1) & (2) have <u>no simultaneous solution</u>.

Summary: 2 linear equations in 2 unknowns represent the intersection of 2 lines in the plane. There are 3 possible cases:

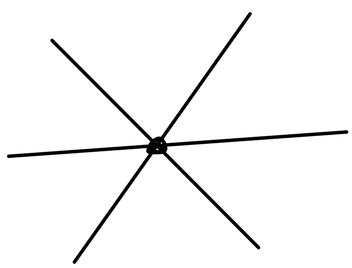


Example of ∞ solutions:

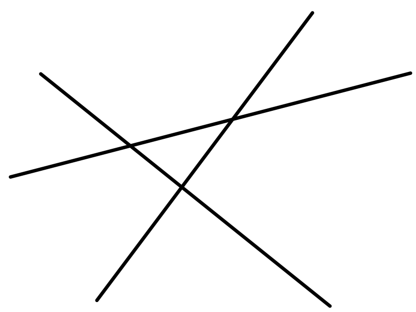
$$\begin{cases} x + y = 1, & (1) \\ 2x + 2y = 2. & (2) \end{cases}$$

In this case, (1) & (2) are secretly the same equation/line twice.

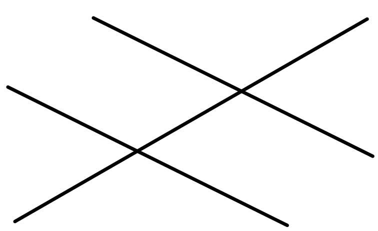
What about 3 linear equations in 2 unknowns, i.e., 3 lines in the plane?



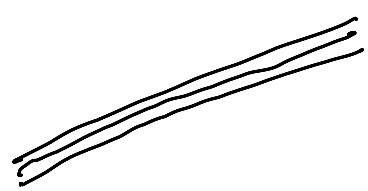
1 solution



0 solutions



0 solutions

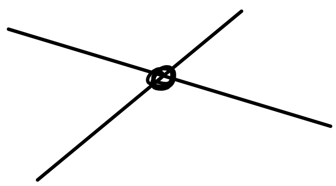


∞ solutions

Some of these cases are UNLIKELY.

2 eqns & 2 unknowns :

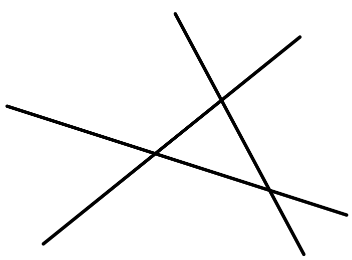
Most likely case is



1 solution

3 eqns & 2 unknowns :

Most likely case is



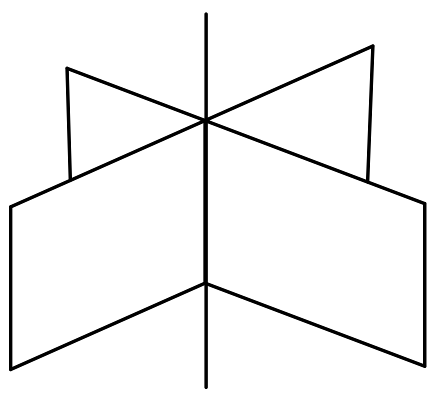
0 solutions

$m \geq 3$ eqns & 2 unknowns :

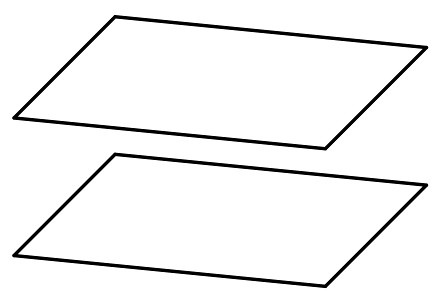
Most likely has NO SOLUTION.

Next Case: $n = 3$ unknowns.

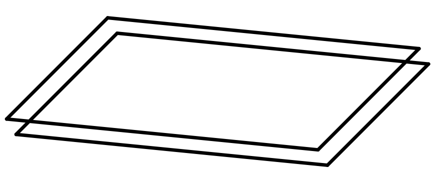
2 linear eqns & 3 unknowns represents the intersection of 2 planes in 3D space. What do we expect?



∞ solutions form a <u>Line</u>



parallel planes mean 0 solutions



identical planes means ∞ solutions forming a plane.

In this case it is impossible to have 1 unique solution.

Example:

$$\begin{aligned} &(1) \\ &(2) \end{aligned} \begin{cases} x + y + z = 0 \\ x + 2y + 3z = 0 \end{cases}$$

First we try to eliminate the x term:

$$(3) = (2) - (1)$$

$$\begin{array}{rl} (2) & x + 2y + 3z = 0 \\ -\ (1) & x + y + z = 0 \\ \hline (3) & 0 + y + 2z = 0 \end{array}$$

Now what?

[Remkark: (3) is still the equation of a plane living in Cartesian x, y, z - space.]

We do not have enough equations to eliminate the y term from (3).

But, we <u>can</u> use (3) to simplify (1) a little bit:

$$\begin{array}{rccccccc}
(1) & x & + & y & + & z & = & 0 \\
-\ (3) & 0 & + & y & + & 2z & = & 0 \\
\hline
(4) & x & + & 0 & - & z & = & 0.
\end{array}$$

Now we have an equation with no x term:

$$(3) \quad 0 + y + 2z = 0$$

and an equation with no y term:

$$(4) \quad x + 0 - z = 0.$$

That's the best we can do!

The important fact is that

the original system 1 & 2 is equivalent to the simpler system of 3 & 4 :

$$\left\{\begin{array}{l} x+y+z=0 \\ x+2y+3z=0 \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{rcr} x & & -z=0 \\ & y & +2z=0 \end{array}\right\}$$

Next time I'll show you how to read off the solution from this.