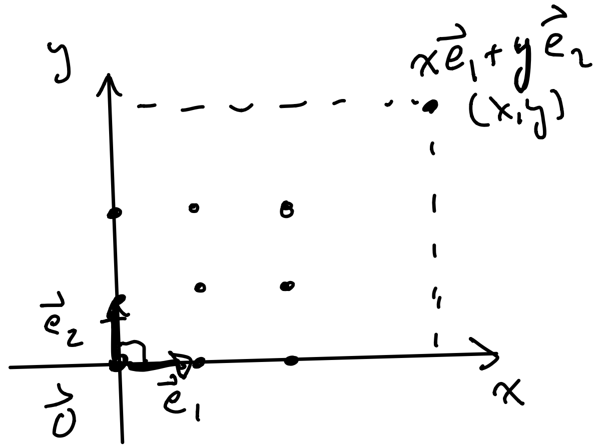
Quiz 1: Beginning of class Tuesday.
(No class Monday.)

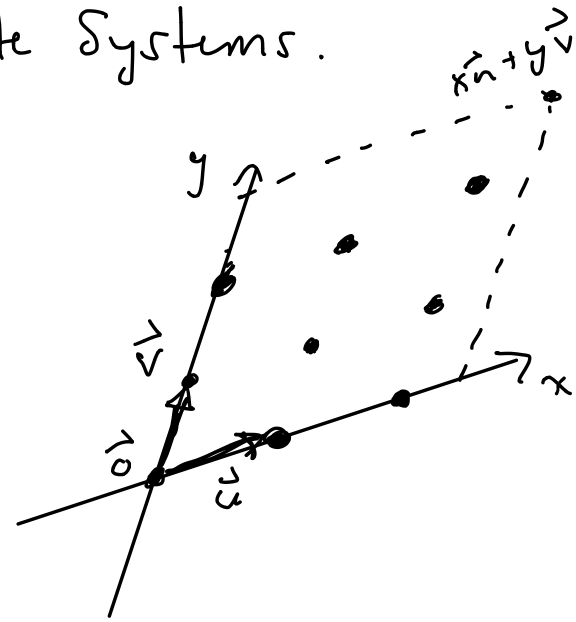
Today: HW1 Discussion.

Problem 1: Coordinate Systems.




$\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ & $\vec{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

standard basis




$\vec{u} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ & $\vec{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

non-standard basis

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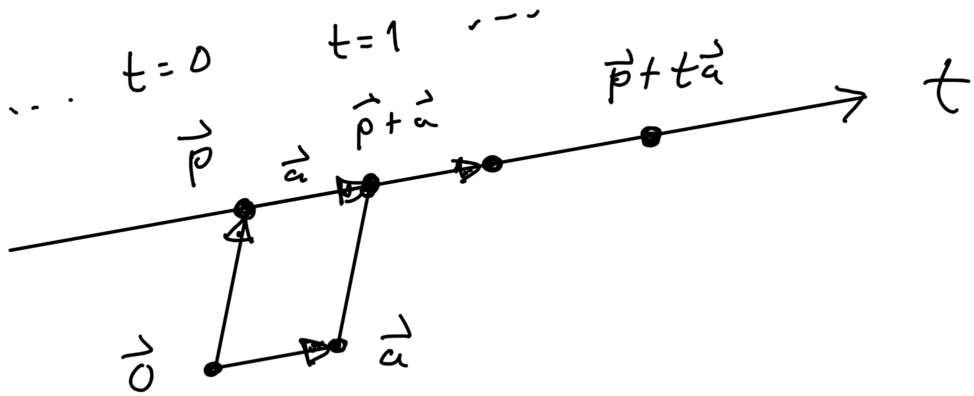
How to describe a line in 2D:

Old way: slope & intercept.

New way: points & vectors.

(1) Point $\vec{p}$ & Direction vector $\vec{a}$.

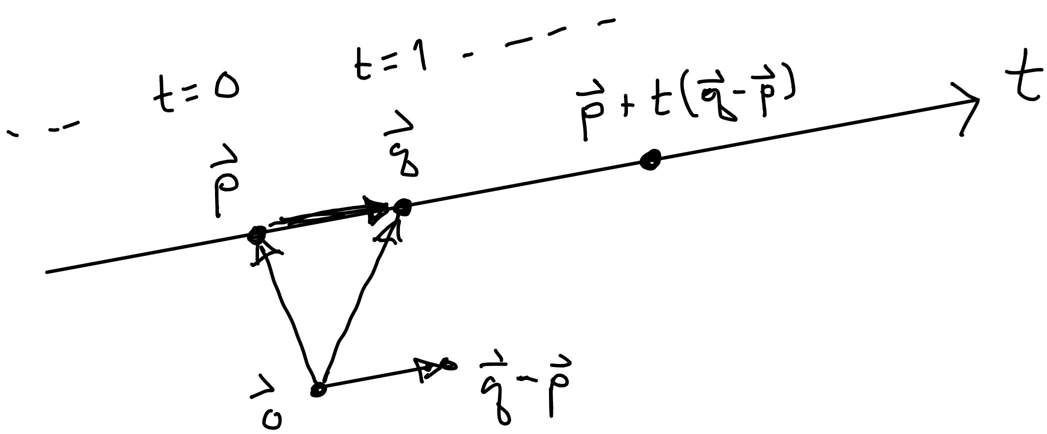



Let $t$ be a parameter. The general point on the Line is $\vec{p}+t\vec{a}$.

$$\text{Line} = \{\vec{p}+t\vec{a} : t \in \mathbb{R}\}$$

= "the set of points $\vec{p}+t\vec{a}$ where $t$ is any real number"

② Two Points $\vec{p}$ & $\vec{q}$.



$$\text{Line} = \{ \vec{p} + t(\vec{q} - \vec{p}) : t \in \mathbb{R} \}$$

$$= \{ (1-t)\vec{p} + t\vec{q} : t \in \mathbb{R} \}$$

$$= \{ x\vec{p} + y\vec{q} : x + y = 1 \}$$

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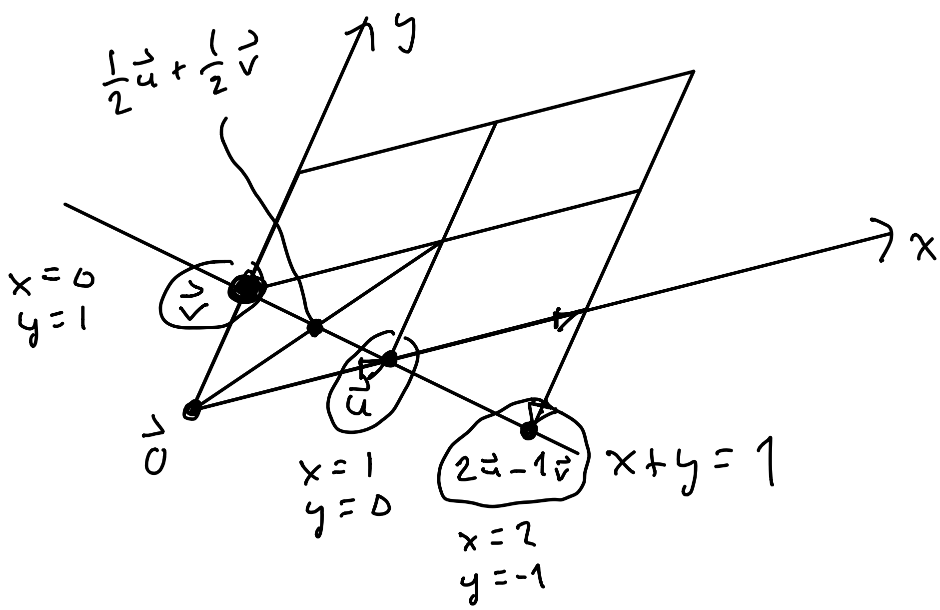
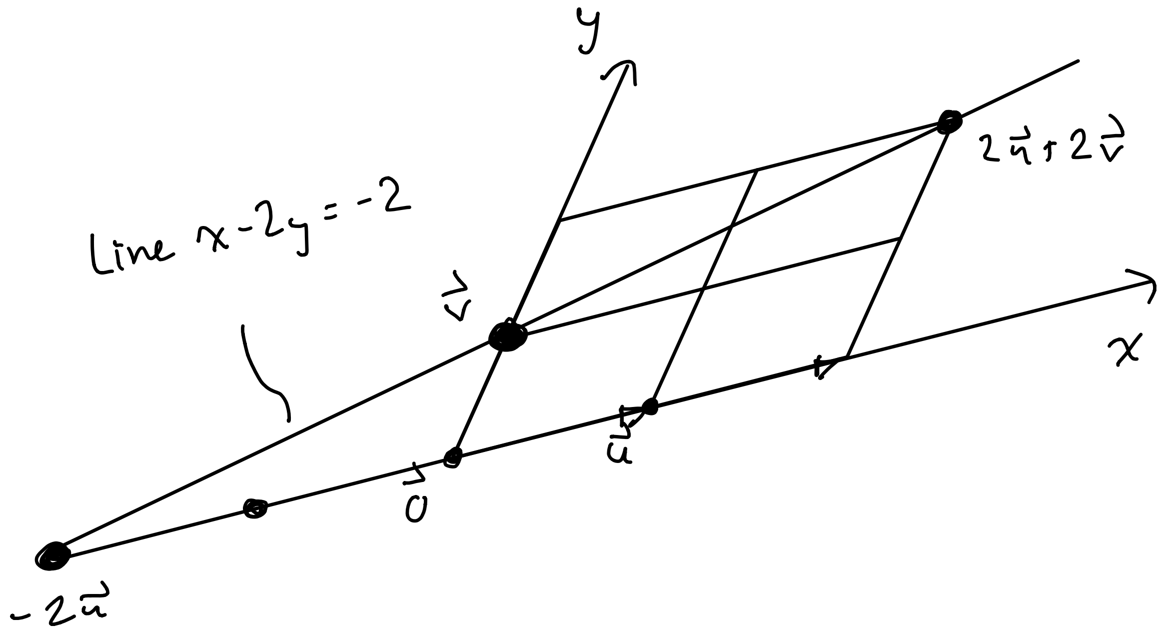
Apply this to Problem 2:




What about $\{ x\vec{u} + y\vec{v} : x - 2y = -2 \}$ ?

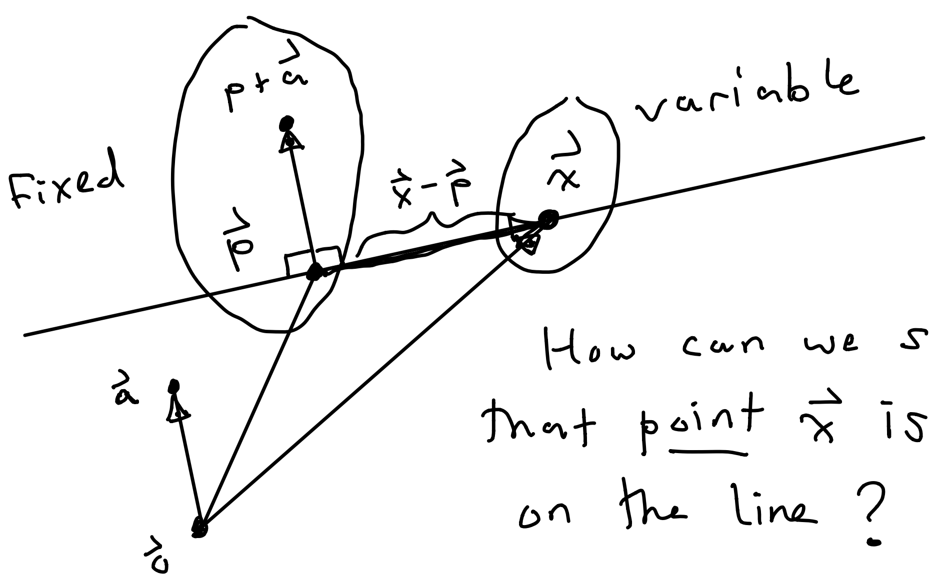
Find two points: $\left. \begin{array}{l} x = 0 \\ y = 1 \end{array} \right\} \; 0\vec{u} + 1\vec{v}.$

$\left. \begin{array}{l} x = -2 \\ y = 0 \end{array} \right\} \; -2\vec{u} + 0\vec{v}$




③rd way to describe a line in 2D:

A point $\vec{p}$ & a "normal vector" $\vec{a}$
perpendicular




How can we say that point $\vec{x}$ is on the line?

point $\vec{x}$ is on the line

$\iff$ $\vec{x}-\vec{p}$ is $\perp$ to $\vec{a}$.

$$\vec{a}\cdot(\vec{x}-\vec{p})=0$$

Arithmetic:

$$\vec{a}\cdot\vec{x}-\vec{a}\cdot\vec{p}=0$$

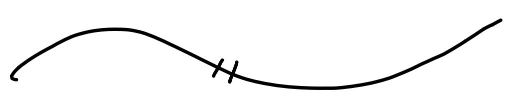
$$\vec{a}\cdot\vec{x}=\vec{a}\cdot\vec{p}$$

given

variable point on the line.

given

Remark: This is an equation of the line, not a parametrization



Higher Dimensions?

The Point-Parallel Vector & Two Point parametrization still work in any number of dimensions.

$\{\vec{p} + t\vec{a} : t \in \mathbb{R}\}$ is a <u>line</u>

$\{x\vec{p} + y\vec{q} : x + y = 1\}$

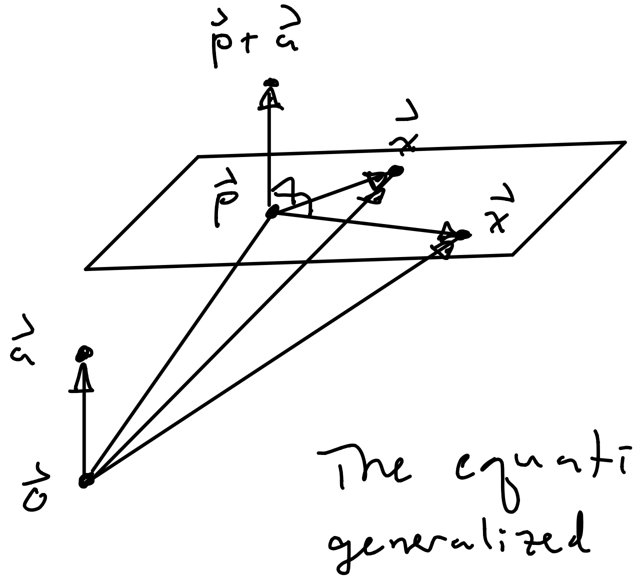
$= \{\vec{p} + t(\vec{q} - \vec{p}) : t \in \mathbb{R}\}$ is a <u>line</u>.

However, the equation

$$\vec{a} \cdot \vec{x} = \vec{a} \cdot \vec{p}$$

is <u>not</u> a line in higher dimensions!

What is it?




For any point $\vec{x}$ in the plane we have

$$\vec{a} \cdot (\vec{x} - \vec{p}) = 0.$$

The equation of a generalized plane.

We call it a "hyperplane."
Later we will see that a hyperplane living in $\mathbb{R}^n$ is an $(n-1)$ dimensional shape.

The Linear Equation

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b$$

$$\vec{a} \cdot \vec{x} = b$$

describes a hyperplane in $\mathbb{R}^n$.

We can't draw this in general, but what can we learn about it?

- It is a "Flat" shape.

Given two points $\vec{x}$ & $\vec{y}$ on the hyperplane, every point of the line $s\vec{x} + t\vec{y}$ with $s+t=1$ is on the hyperplane.

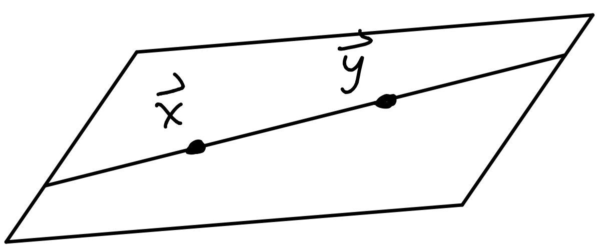
Proof: $\vec{x}$ & $\vec{y}$ on the hyperplane

$\Longrightarrow \quad \vec{a} \cdot \vec{x} = b \quad \& \quad \vec{a} \cdot \vec{y} = b$

$$\Longrightarrow \quad \vec{a} \cdot (s\vec{x} + t\vec{y})$$
$$= s(\vec{a} \cdot \vec{x}) + t(\vec{a} \cdot \vec{y})$$
$$= sb + tb$$
$$= (\underbrace{\cancel{s+t}}_{1})\, b = b.$$

$\Longrightarrow$ the point $s\vec{x} + t\vec{y}$ is also on the hyperplane. ///

e.g. $\vec{x}$ & $\vec{y}$ on a 2D plane:




the whole line is on the plane.

- The hyperplane $\vec{a} \cdot \vec{x} = b$ is perpendicular to the vector $\vec{a}$.

<u>Proof</u>: For any points $\vec{x}$ & $\vec{y}$ on the hyperplane we have

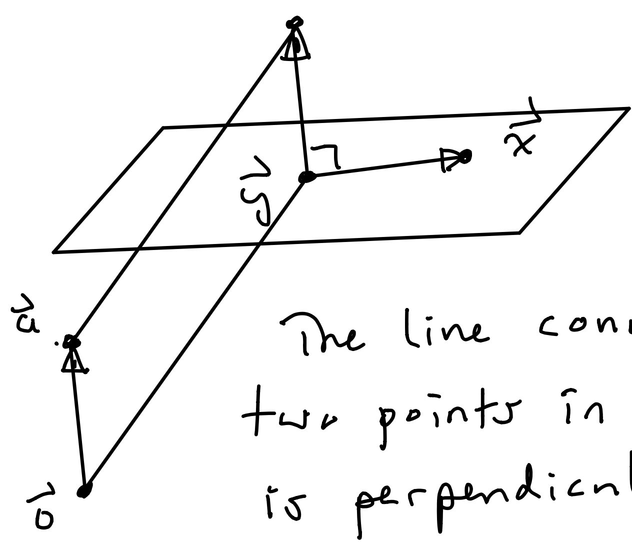
$$\vec{a}\cdot\vec{x} = b \quad \& \quad \vec{a}\cdot\vec{y} = b.$$

But then we also have

$$\begin{aligned}\vec{a}\cdot(\vec{x}-\vec{y}) &= \vec{a}\cdot\vec{x} - \vec{a}\cdot\vec{y} \\ &= b - b \\ &= 0,\end{aligned}$$

hence $\vec{a}$ is perpendicular to $\vec{x}-\vec{y}$.

Picture:




The line connecting any two points in the hyperplane is perpendicular to $\vec{a}$.

- The equation $\vec{a}\cdot\vec{x} = b$ does not explicity tell us any point on the hyperplane. Can we find a point?

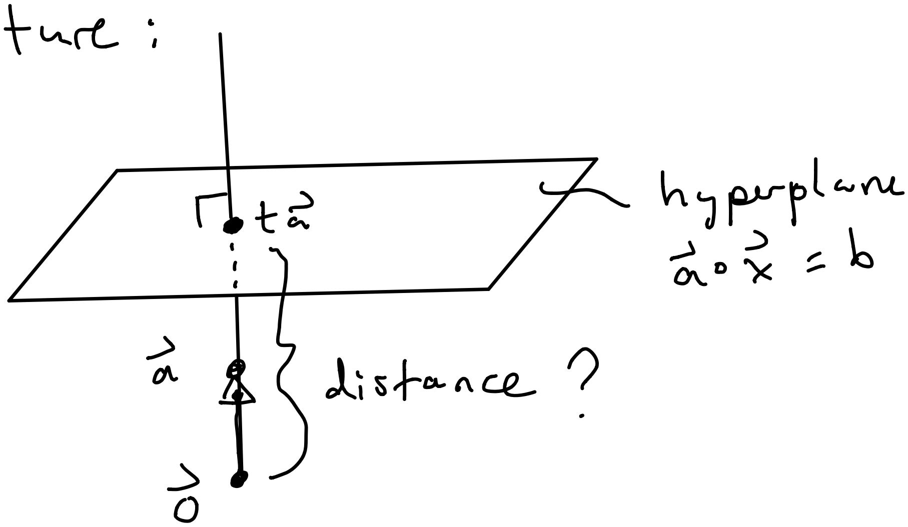
Claim: The point $\vec{x} = \frac{b}{\|\vec{a}\|^2}\vec{a}$ is on the hyperplane.

<u>Proof</u>: $\vec{a} \cdot \left( \frac{b}{\|\vec{a}\|^2} \vec{a} \right)$

scalar

$$= \frac{b}{\|\vec{a}\|^2}(\vec{a} \cdot \vec{a}) = \frac{b}{\cancel{\|\vec{a}\|^2}} \cancel{\|\vec{a}\|^2} = b \checkmark$$

Picture:



Some point of the form $t\vec{a}$ is on the hyperplane. Find $t$:

$$\vec{a} \cdot (t\vec{a}) = b$$

$$t(\vec{a} \cdot \vec{a}) = b$$

$$t\,\|\vec{a}\|^2 = b$$

$$t = b/\|\vec{a}\|^2 .$$

What is the length of the vector $t\vec{a}$?

$$\|t\vec{a}\| = ?$$

Claim: $\|t\vec{a}\| = |t|\,\|\vec{a}\|$

Proof:

$$\|t\vec{a}\|^2 = \|(ta_1, ta_2, \ldots, ta_n)\|^2$$

$$= (ta_1)^2 + (ta_2)^2 + \cdots + (ta_n)^2$$

$$= t^2(a_1^2 + a_2^2 + \cdots + a_n^2)$$

$$= t^2\|\vec{a}\|^2 .$$

Take (positive) square roots:

$$\|t\vec{a}\|^2 = t^2\|\vec{a}\|^2$$

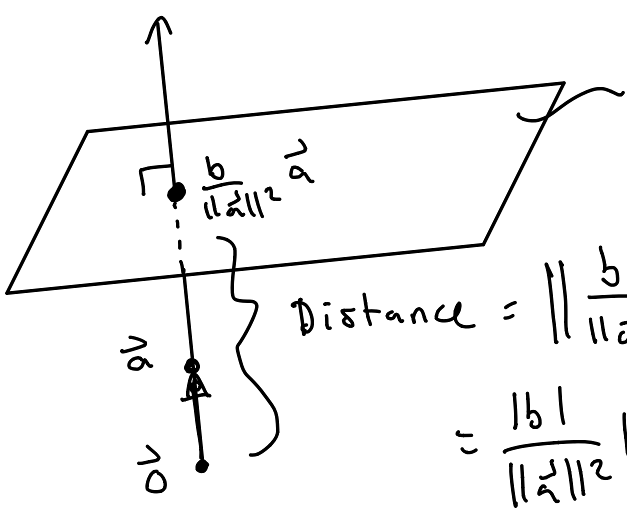
$$\|t\vec{a}\| = {}^{+}\!\sqrt{t^2}\,\|\vec{a}\|$$

$$\|t\vec{a}\| = |t|\,\|\vec{a}\| \quad \checkmark$$

[Application: For any vector $\vec{v}$, the vector $\frac{1}{\|\vec{v}\|}\vec{v}$ has length 1. Indeed,

$$\left\|\frac{1}{\|\vec{v}\|}\vec{v}\right\| = \left|\frac{1}{\cancel{\|\vec{v}\|}}\right|\cancel{\|\vec{v}\|} = 1 \quad \checkmark$$ ]

Back to Our Picture:

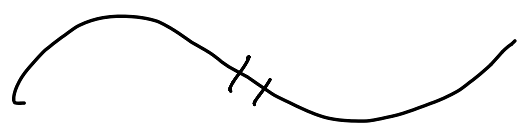



hyperplane
$\vec{a}\cdot\vec{x} = b$

$$\text{Distance} = \left\|\frac{b}{\|\vec{a}\|^2}\vec{a}\right\|$$

$$= \frac{|b|}{\|\vec{a}\|^2}\|\vec{a}\| = \frac{|b|}{\|\vec{a}\|}.$$

Summarize: The equation $\vec{a} \cdot \vec{x} = b$ represents a <u>flat</u> shape that is <u>perpendicular</u> to the vector $\vec{a}$ and has <u>distance</u> $|b| / \|\vec{a}\|$ from the origin at its closest point.



For the Quiz:

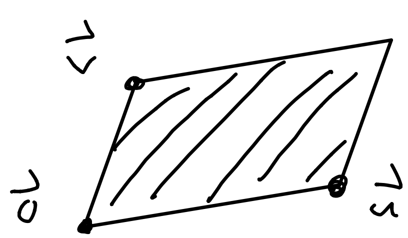
Rules of vector arithmetic,
applied to column vectors
or applied in general.

Pythagorean Theorem:

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos \theta .$$

2D pictures of vector arithmetic

e.g.



Shaded region is

$$\{ x\vec{u} + y\vec{v} : 0 \leq x \leq 1 \ \& \ 0 \leq y \leq 1 \}.$$

In particular, be able to draw the lines: $\vec{p} + t\vec{a}$

or $\vec{p} + t(\vec{q} - \vec{p})$

or $\vec{a} \cdot \vec{x} = \vec{a} \cdot \vec{p}$.