HW 1 due <u>before</u> tomorrow's class.

We will discuss solutions in class.

Quiz 1 beginning of class <u>Tues</u> May 26.

The concept of an abstract "vector space."

A set V of vectors.

3 operations:

$$\vec{u}, \vec{v} \in V \Rightarrow \vec{u} + \vec{v} \in V$$

$$a \in \mathbb{R}, \vec{v} \in V \Rightarrow a\vec{v} \in V$$

$$\vec{u}, \vec{v} \in V \Rightarrow \vec{u} \bullet \vec{v} \in \mathbb{R}.$$

These 3 operations must satisfy a bunch of obvious rules.

Example: $V = \mathbb{R}^n$

$=$ column vectors with n real entries.

This vector space has a "standard basis of vectors"

$$\vec{e}_1 = (1, 0, 0, \ldots, 0)$$
$$\vec{e}_2 = (0, 1, 0, \ldots, 0)$$
$$\vdots$$
$$\vec{e}_n = (0, 0, \ldots, 0, 1)$$

Every vector $\vec{v} = (v_1, v_2, \ldots, v_n)$ has a <u>unique expression</u> as a "linear combination" of basis vectors:

$$\begin{aligned}
\vec{v} &= (v_1, v_2, \ldots, v_n) \\
&= v_1 (1, 0, \ldots, 0) \\
&\quad + v_2 (0, 1, 0, \ldots, 0) \\
&\quad \vdots \\
&\quad + v_n (0, 0, \ldots, 0, 1) \\
&= v_1 \vec{e}_1 + v_2 \vec{e}_2 + \cdots + v_n \vec{e}_n .
\end{aligned}$$

Fancy!

In fact, we say $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$ are an "ortho-normal basis" because

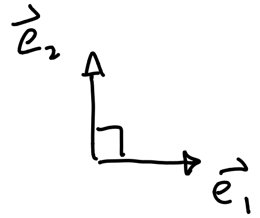
$\vec{e}_i \cdot \vec{e}_i = 1^2 + 0^2 + \cdots + 0^2 = 1$ $\qquad \|\vec{e}_i\| = 1$

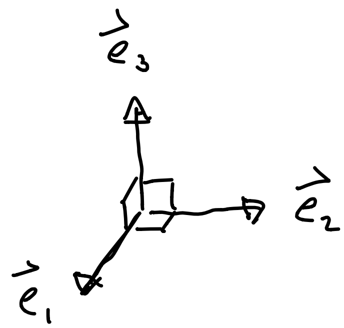
$\vec{e}_i \cdot \vec{e}_j = 0 + 0 + \cdots + 0 = 0$ $\qquad \vec{e}_i \perp \vec{e}_j$

$(i \neq j)$

i.e., each basis vector is a "normalized" vector (has length 1) and any two basis vectors are "orthogonal" (pependicular).

Picture:

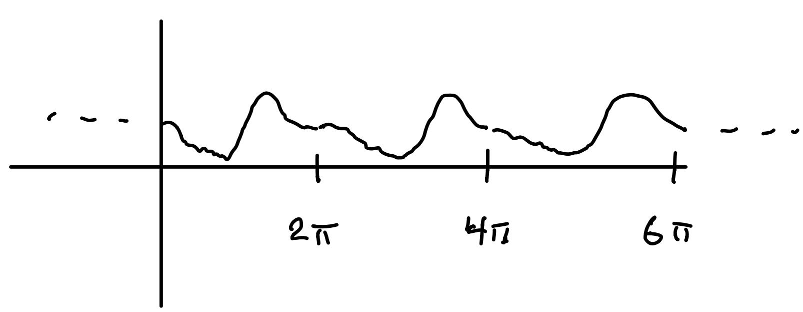




"Cartesian Coordinates"

Why bother with this abstraction?

Reason: There exist exotic vector spaces. Example: Consider a periodic function $f: \mathbb{R} \to \mathbb{R}$ and $f(x+2\pi) = f(x)$.



The set of periodic functions is an abstract "vector space"!

3 operations:

$$f(x),\ g(x) \implies (f+g)(x) = f(x) + g(x)$$

$$a \in \mathbb{R},\ f(x) \implies (af)(x) = a f(x)$$

What about "dot product"?

$$\underset{\text{functions}}{f(x) \bullet g(x)} = \int_0^{2\pi} f(t) g(t)\, dt \in \mathbb{R}$$

number

I claim that these 3 operations satisfy all the rules of a vector space.

Missing Piece: There is no obvious choice of "standard basis vectors."

Theorem (Fourier, Dirichlet):

The set of functions

$$\frac{1}{\sqrt{2\pi}},\ \frac{\sin(x)}{\sqrt{\pi}},\ \frac{\sin(2x)}{\sqrt{\pi}},\ \ldots$$

$$\frac{\cos(x)}{\sqrt{\pi}},\ \frac{\cos(2x)}{\sqrt{\pi}},\ \ldots$$

is an ortho-normal basis.

This means every periodic function $f(x)$ (e.g., a musical sound) has a unique expression:

$$f(x) = a_0 + a_1 \sin(x) + a_2 \sin(2x) + \cdots$$
$$+ b_1 \cos(x) + b_2 \cos(2x) + \cdots$$

Any periodic signal is an (infinite) sum of pure sine waves.

The "Fourier Series"

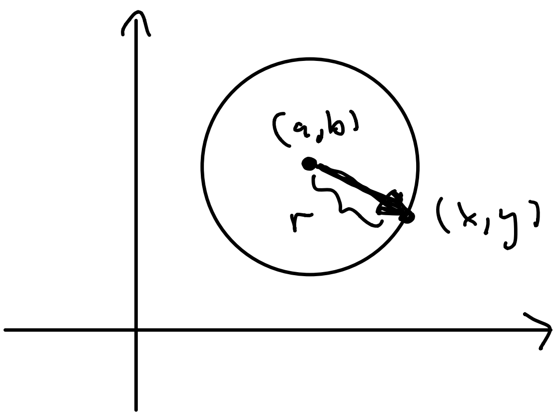
Music synthesizer chooses the "Fourier coefficients" $a_0, a_1, a_2, \ldots, b_1, b_2, \ldots$ to create different kinds of sounds.

Next Topic:

Systems of Linear Equations

"Analytic Geometry" associates shapes to equations.

Example: The equation of a circle in the Cartesian plane $\mathbb{R}^2$.



Let (x,y) be a general point on the circle.

Then (x,y) satisfies an equation:

Distance between (x,y) & (a,b) is r.

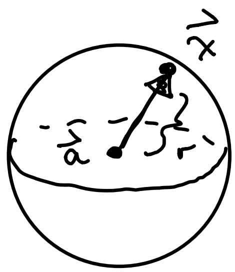
Lenth of vector $(x-a, y-b)$ is r.

$$\|(x-a, y-b)\| = r$$

$$\sqrt{(x-a)^2 + (y-b)^2} = r$$

$$(x-a)^2 + (y-b)^2 = r^2.$$

Equation of a sphere in $\mathbb{R}^3$?



$$\|\vec{x} - \vec{a}\| = r$$

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$$

More generally:

$$(x_1 - a_1)^2 + (x_2 - a_2)^2 + \cdots + (x_n - a_n)^2 = r^2$$

is the equation of the "hypersphere" living in $\mathbb{R}^n$ with center $(a_1, a_2, \cdots, a_n)$ and radius r.

We can no longer draw the picture, but we can still do computations.

But in this class we are interested in so-called Linear Equations of the form

$$\boxed{a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b}$$

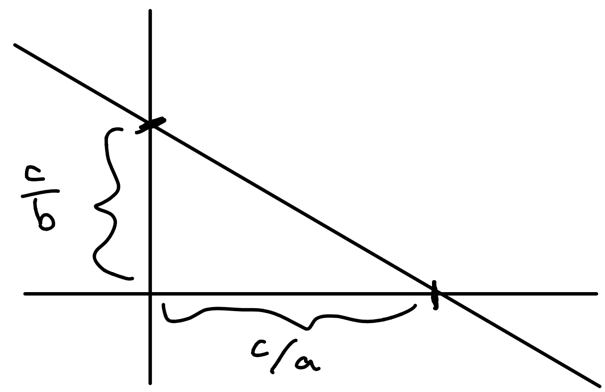
What kind of shape does this represent?

In 2D : $\cancel{a_1 x_1 + a_2 x_2 = b}$

$ax + by = c$.

If $b \neq 0$,

$y = -\frac{a}{b}x + \frac{c}{b}$



Equation of a Line in $\mathbb{R}^2$.

But there is a better way to think about this equation.

$$ax + by = c$$

$$(a,b) \cdot (x,y) = c$$

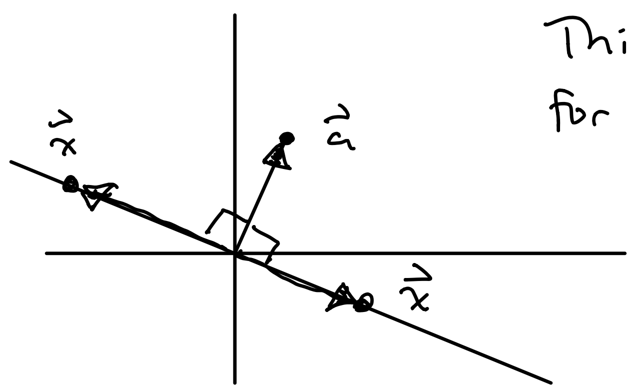
$$\vec{a} \cdot \vec{x} = c$$

Let's consider the special case $c = 0$.

$$\vec{a} \cdot \vec{x} = 0$$

This equation has direct geometric meaning: $\vec{a}$ & $\vec{x}$ are <u>perpendicular</u>.

We can draw this:



This equation holds for any point $\vec{x}$ on the line.

Conclusion: $\vec{a} \cdot \vec{x} = 0$ is the line (through the origin) that is perpendicular to the vector $\vec{a}$.

Before we discuss $\vec{a} \cdot \vec{x} = c \neq 0$, let's discuss higher dimensions.

$$ax + by + cz = 0$$

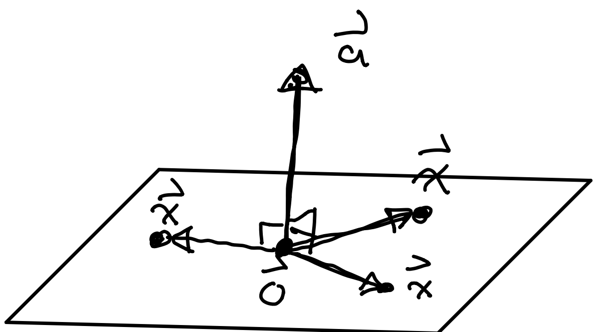
What shape does this describe?

I claim that this describes a **plane** in Cartesian space $\mathbb{R}^3$.

$$ax + by + cz = 0$$

$$(a, b, c) \cdot (x, y, z) = 0$$

$$\vec{a} \cdot \vec{x} = 0 \qquad (\text{i.e., } \vec{a} \perp \vec{x}).$$



The set of $\vec{x}$ such that

$$\vec{a} \cdot \vec{x} = 0$$

forms the plane (through the origin) that is perpendicular to the given vector $\vec{a}$.

Wild Question:

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = 0$$

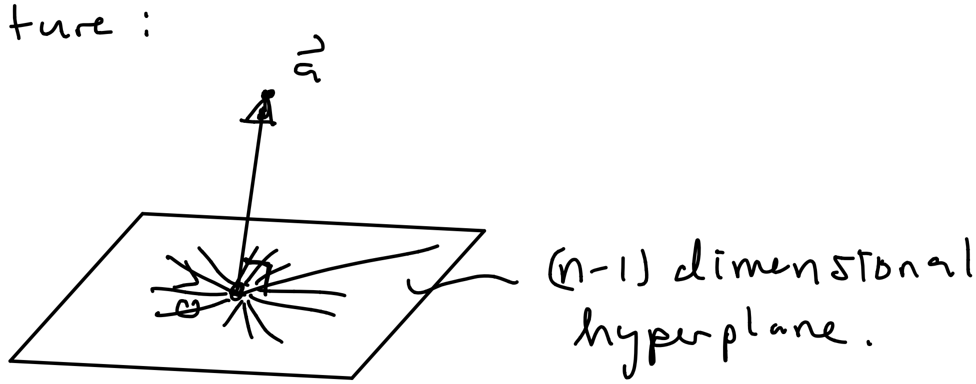
$$(a_1, a_2, \ldots, a_n) \cdot (x_1, \ldots, x_n) = 0$$

$$\vec{a} \cdot \vec{x} = 0$$

Defines a shape in $\mathbb{R}^n$. What is it?

Answer: It is the $(n-1)$ dimensional "hyperplane" (through the origin) that is $\perp$ the given vector $\vec{a}$.

Picture:



Later we will make this rigorous by showing how to find a set of $n-1$ "basis vectors" for the hyperplane.

General Principle:

- One equation in n variables defines an $(n-1)$ dimensional shape.
- Two simultaneous equations in n varicbles (probably) defines an $n-2$ dimensional shape.

Example: Solve for x, y, z

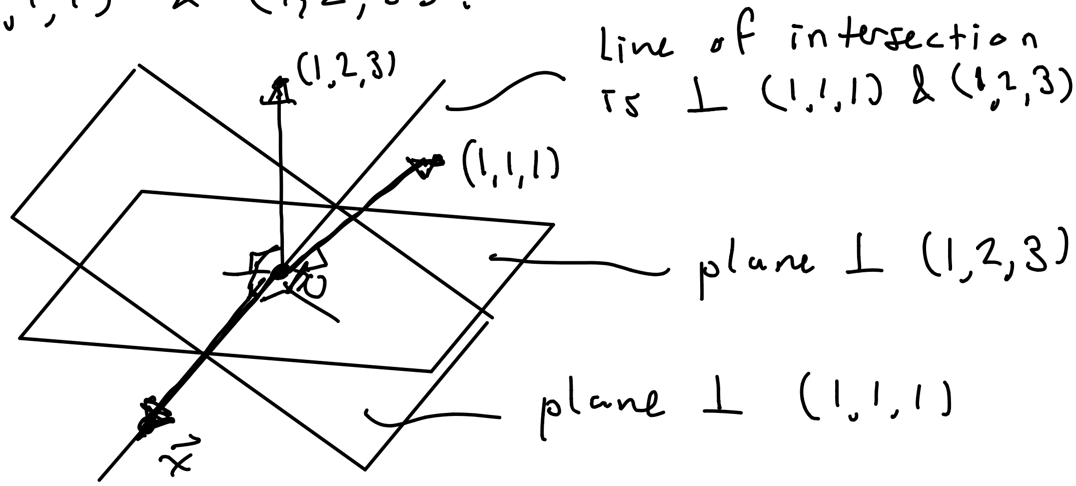
$$\begin{cases} x + y + z = 0 \\ x + 2y + 3z = 0 \end{cases}$$

2 equations in 3 unknowns. should define a 1-dimensional shape living in Cartesian space $\mathbb{R}^3$.

What could it be?

$$\begin{cases} (1,1,1) \cdot (x,y,z) = 0 \\ (1,2,3) \cdot (x,y,z) = 0. \end{cases}$$

means that $\vec{x} = (x,y,z)$ is simultaneously $\perp$ to both $(1,1,1)$ & $(1,2,3)$.



$$\begin{cases} x + y + z = 0 \\ x + 2y + 3z = 0 \end{cases}$$

$\iff$ (x,y,z) is on the Line (through the origin) that is simultaneously $\perp$ to $(1,1,1)$ & $(1,2,3)$

$\iff$ (x,y,z) is on the intersection of the planes $x+y+z=0$ & $x+2y+3z=0$.

Can we be more explicit?

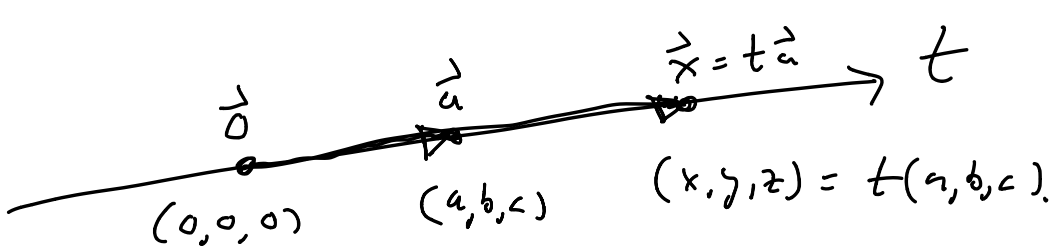
$x = ?$

$y = ?$

$z = ?$

How can we describe a line in 3D?

We need a parameter (call it "t").



Line through the origin has the form $(x, y, z) = t(a, b, c)$

$$\begin{cases} x = ta \\ y = tb \\ z = tc \end{cases}$$ for some "slopes" a, b, c.

We just need to find the "slopes" a, b, c . . . .

NEXT WEEK.