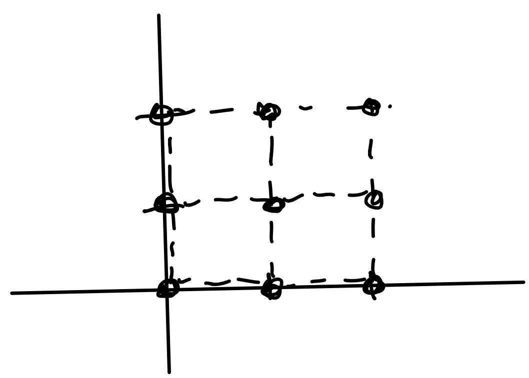
HW1 due Friday <u>before</u> class.

Hint for 1(a):

Draw the 9 points (x, y) where

$x, y \in \{0, 1, 2\}$

"x and y are members of the set 0,1,2."

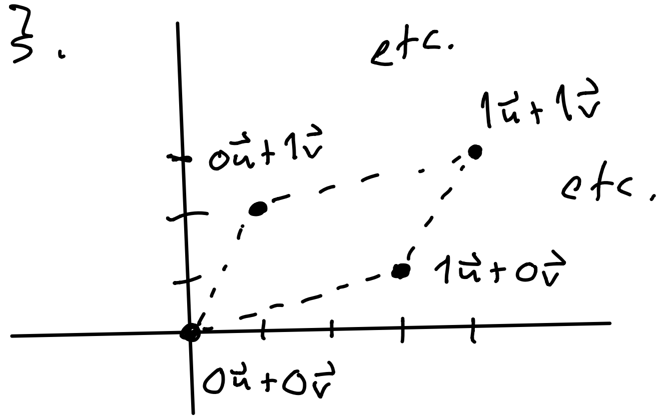


1(c): Let $\vec{u} = (3,1)$ & $\vec{v} = (1,2)$.

Draw the 9 points $x\vec{u} + y\vec{v}$ where

$x, y \in \{0, 1, 2\}$.

$$\vec{u} + \vec{v} = \begin{pmatrix} 3+1 \\ 1+2 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$



Review of Vector Arithmetic:

Three operations,

$\vec{u}, \vec{v} \in \mathbb{R}^n \Rightarrow \vec{u} + \vec{v} \in \mathbb{R}^n$ addition

$a \in \mathbb{R}, \vec{u} \in \mathbb{R}^n \Longrightarrow a\vec{u} \in \mathbb{R}^n$ scalar mult.

$\vec{u}, \vec{v} \in \mathbb{R}^n \Longrightarrow \vec{u} \bullet \vec{v} \in \mathbb{R}$ dot product

vector + vector = vector

scalar × vector = vector

vector • vector = scalar

What about

vector × vector = vector ?

In general, there is no sensible way to "multiply vectors."

However in 2D & 3D there are some peculiar special cases:

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \times \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} u_1 v_1 - u_2 v_2 \\ u_1 v_2 + u_2 v_1 \end{pmatrix}$$

[Think:

$(u_1 + u_2\sqrt{-1})(v_1 + v_2\sqrt{-1})$

$= (u_1v_1 - u_2v_2) + (u_1v_2 + u_2v_1)\sqrt{-1}$]

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \times \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{pmatrix}$$

STRANGE...

This is called the "cross product." We will discuss it later. It *only* applies to 3D vectors.

These operations satisfy many obvious-looking rules:

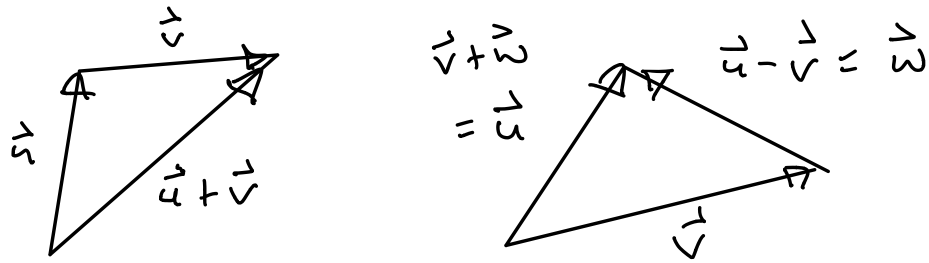
$\vec{u} + \vec{v} = \vec{v} + \vec{u}$

$\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$

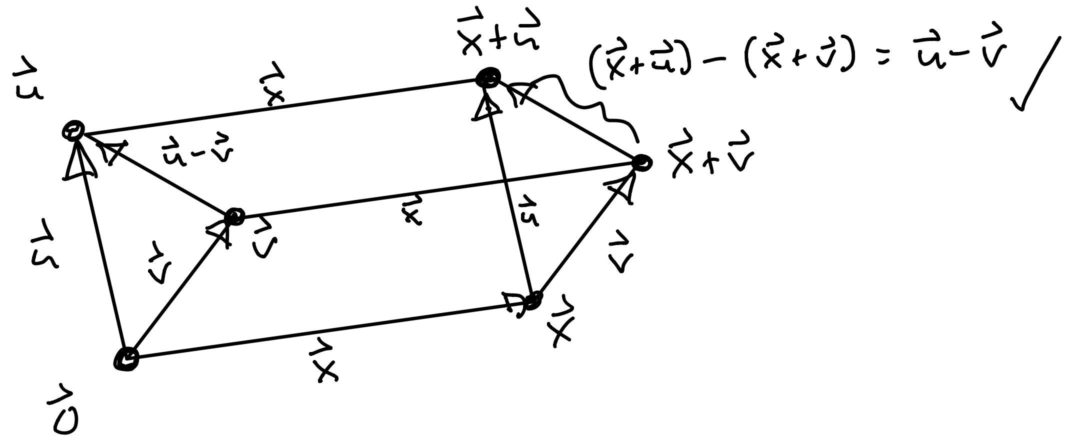
$\vdots$

etc.

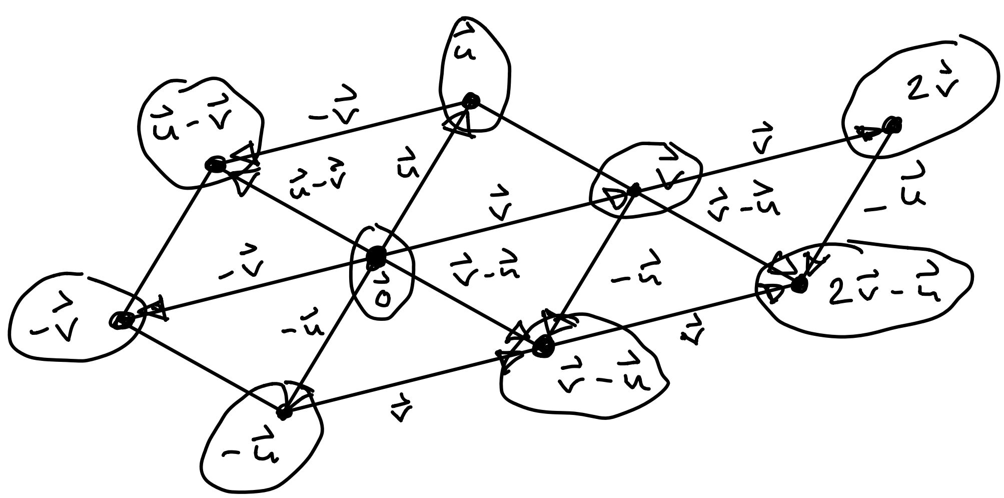
But they also have geometric meaning.



Mnemonic: vector = head minus tail



Note that everything fits together:



Geometric meaning of the dot product is the Pythagorean Theorem.

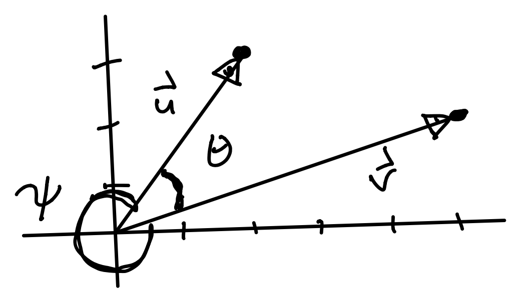
$$\|\vec{u}\|^2 = \vec{u}\cdot\vec{u} = u_1^2 + u_2^2 + \cdots + u_n^2$$

$$\vec{u}\cdot\vec{v} = \|\vec{u}\|\,\|\vec{v}\|\cos\theta$$

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|} = \frac{\vec{u}\cdot\vec{v}}{\sqrt{\vec{u}\cdot\vec{u}}\,\sqrt{\vec{v}\cdot\vec{v}}}$$

Examples:

2D: $\vec{u} = (\underset{u_1}{2}, \underset{u_2}{3})$ & $\vec{v} = (\underset{v_1}{5}, \underset{v_2}{2})$



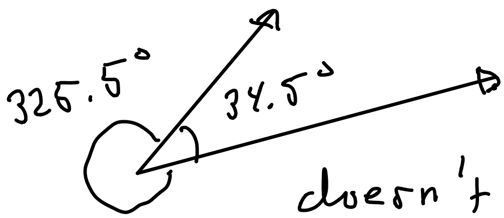
compute the angle θ. (or ψ)

Just need the dot products:

$$\vec{u}\cdot\vec{v} = u_1v_1 + u_2v_2 = 10 + 6 = 16$$
$$\vec{u}\cdot\vec{u} = u_1u_1 + u_2u_2 = 4 + 9 = 13$$
$$\vec{v}\cdot\vec{v} = v_1v_1 + v_2v_2 = 25 + 4 = 29.$$

$$\cos\theta = \frac{\vec{u}\circ\vec{v}}{\sqrt{\vec{u}\circ\vec{u}}\sqrt{\vec{v}\circ\vec{v}}} = \frac{16}{\sqrt{13}\sqrt{29}} = 0.824$$

$$\Longrightarrow \quad \theta = \arccos(0.824)$$
$$= 34.5^\circ \text{ or } 325.5^\circ.$$
$$-34.5^\circ$$



doesn't matter.

1D: Let's make sure we understand the formula $\vec{u}\circ\vec{v} = \|\vec{u}\|\|\vec{v}\|\cos\theta$ in 1-dimensional space.

$\vec{u} = (u)$ 1D vectors are just scalars.
$\vec{v} = (v)$

$$\|\vec{u}\|^2 = \vec{u}\circ\vec{u} = u^2$$

$$\|\vec{u}\| = \sqrt{u^2} = |u| \quad \text{absolute value}$$

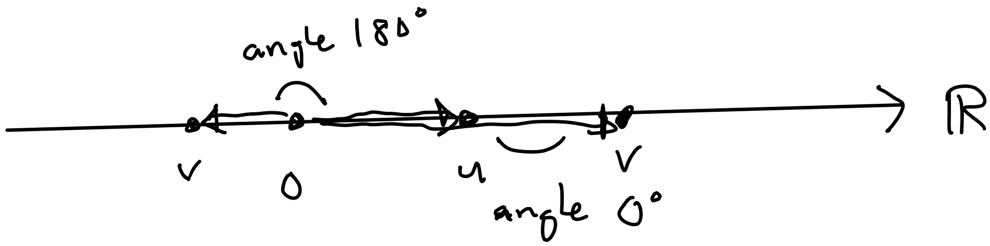
$$= \begin{cases} u & \text{if } u \geq 0, \\ -u & \text{if } u < 0. \end{cases}$$

The formula $\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos\theta$ becomes

$$uv = |u||v| \cos\theta .$$

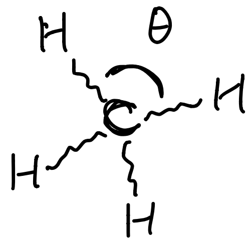
Does this make any sense?

$$uv = \begin{cases} |u||v| & \text{if } u \,\&\, v \text{ have same sign} \\ -|u||v| & \text{if } u \,\&\, v \text{ opposite sign.} \end{cases}$$



3D: The tetrahedral angle.

Methane molecule

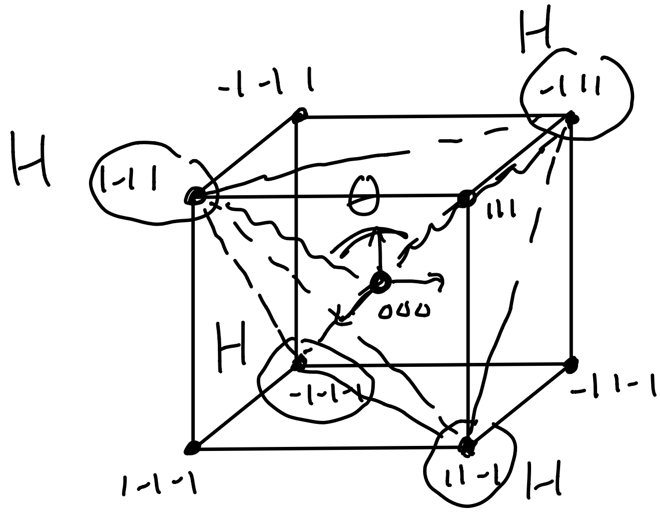


Angle between two Hydrogen atoms is

$$\theta \approx 109.5°$$

WHY?

To compute this we will introduce a coordinate system:



$(\pm 1, \pm 1, \pm 1)$ are the vertices of a cube centered at $(0,0,0)$

Half of the vertices form a regular tetrahedron.

Pick any two Hydrogen atoms, say

$(1,-1,1)$ & $(-1,1,1)$.

Now it's just mindless calculation:

$$\cos\theta = \frac{(1,-1,1)\cdot(-1,1,1)}{\sqrt{(1,-1,1)\cdot(1,-1,1)}\ \sqrt{(-1,1,1)\cdot(-1,1,1)}}$$

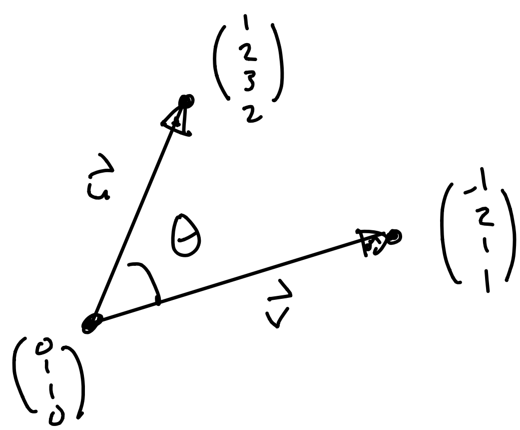
$$= \frac{-1-1+1}{\sqrt{1+1+1}\ \sqrt{1+1+1}} = \frac{-1}{\sqrt{3}\sqrt{3}} = -\frac{1}{3}$$

$$\Longrightarrow \quad \cos\theta = -1/3$$

$$\theta = \arccos(-1/3)$$

$$\approx 109.4712206\ldots \text{ degrees.}$$

4D: Compute the following angle:



Observe:

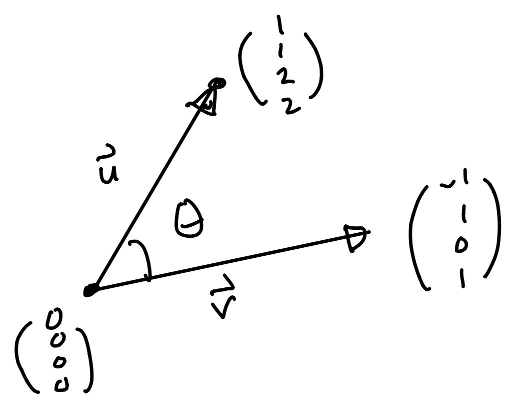
$\vec{u}$ = head − tail

$= (1,2,3,2) - (0,1,1,0) = (1,1,2,2)$

$\vec{v}$ = head − tail

$= (-1,2,1,1) - (0,1,1,0) = (-1,1,0,1)$

The picture in standard position:



Mindless computation:

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\sqrt{\vec{u}\cdot\vec{u}}\,\sqrt{\vec{v}\cdot\vec{v}}} = \frac{-1+1+0+2}{\sqrt{1+1+4+4}\,\sqrt{1+1+0+1}}$$

$$= \frac{2}{\sqrt{10}\,\sqrt{3}} = 0.365$$

$$\theta = \arccos(0.365)$$
$$\approx 68.6^\circ.$$

[Meaning: The arrows $\vec{u}$ & $\vec{v}$ live inside a 2D plane sitting in 4D space. We measure the angle in the 2D plane.]

100D : Consider any $\vec{x}, \vec{y} \in \mathbb{R}^{100}$ satisfying

$$\vec{x}\cdot\vec{y} = 0 \quad \& \quad \vec{x}\cdot\vec{x} = \vec{y}\cdot\vec{y} = 1.$$

Compute the angle between

$$\vec{u} = 2\vec{x} + 3\vec{y}$$
$$\vec{v} = 5\vec{x} + 2\vec{y}.$$

How on Earth ?!

Mindlessly apply the arithmetic:

$$\begin{aligned}
\vec{u}\cdot\vec{v} &= (2\vec{x}+3\vec{y})\cdot(5\vec{x}+2\vec{y}) \\
&= 10\,\vec{x}\cdot\vec{x} + 4\,\vec{x}\cdot\vec{y} + 15\,\vec{y}\cdot\vec{x} + 6\,\vec{y}\cdot\vec{y} \\
&= 10\,\underbrace{\cancel{\vec{x}\cdot\vec{x}}}_{1} + 19\,\underbrace{\cancel{\vec{x}\cdot\vec{y}}}_{0} + 6\,\underbrace{\cancel{\vec{y}\cdot\vec{y}}}_{1} \\
&= 16.
\end{aligned}$$

$$\begin{aligned}
\vec{u}\cdot\vec{u} &= (2\vec{x}+3\vec{y})\cdot(2\vec{x}+3\vec{y}) \\
&= 4\,\underbrace{\cancel{\vec{x}\cdot\vec{x}}}_{1} + 12\,\underbrace{\cancel{\vec{x}\cdot\vec{y}}}_{0} + 9\,\underbrace{\cancel{\vec{y}\cdot\vec{y}}}_{1} \\
&= 13.
\end{aligned}$$

$$\vec{v} \circ \vec{v} = (5\vec{x} + 2\vec{y}) \circ (5\vec{x} + 2\vec{y})$$

$$= 25 \underbrace{\cancel{\vec{x} \circ \vec{x}}}_{1} + 20 \underbrace{\cancel{\vec{x} \circ \vec{y}}}_{0} + 4 \underbrace{\cancel{\vec{y} \circ \vec{y}}}_{1}$$

$$= 29.$$

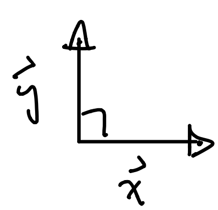
Done.

$$\cos\theta = \frac{\vec{u} \circ \vec{v}}{\sqrt{\vec{u} \circ \vec{u}}\,\sqrt{\vec{v} \circ \vec{v}}} = \frac{16}{\sqrt{13}\,\sqrt{29}} = 0.824$$

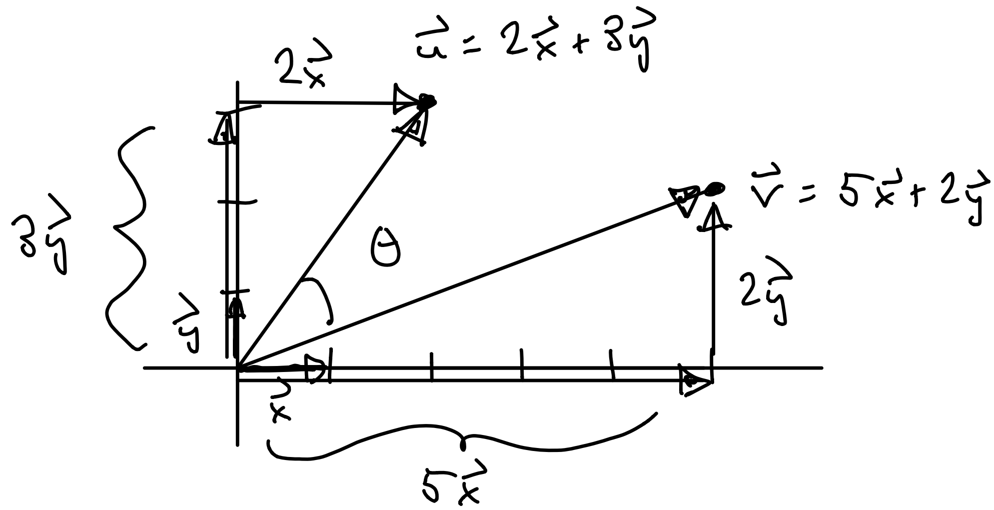
$$\theta = \arccos(0.824) \approx 34.5^\circ.$$

Compare with the 2D example above. Why is it the same?

Facts $\vec{x} \circ \vec{y} = 0$ & $\vec{x} \circ \vec{x} = \vec{y} \circ \vec{y} = 1$ say that $\vec{x}$ & $\vec{y}$ are "perpendicular unit vectors." Even though they live in 100D space, we can draw them:



Then we can draw $\vec{u}$ & $\vec{v}$:



Where have we seen this before?

Big Idea:

We can use the vectors $\vec{x}$ & $\vec{y}$ to define a temporary coordinate system. We can easily compute θ inside this coordinate system.

Say: $\vec{u} = 2\vec{x} + 3\vec{y} =$ "(2, 3)"

$\vec{v} = 5\vec{x} + 2\vec{y} =$ "(5, 2)"

[Remark: However, if the coordinate vectors $\vec{x}$ & $\vec{y}$ are <u>not</u> perpendicular, i.e., if $\vec{x} \cdot \vec{y} \neq 0$, then the angles will be distorted.]