HW 1 due Friday <u>before</u> class.

Office Hours: MTuWTh after class.

This Week: Arithmetic of Vectors.

What is a vector?

Let $\mathbb{R}^n$ be the set of column vectors with n real coordinates

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

$$\vec{0} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$ the <u>zero vector</u>.

We define two algebraic operations:

- Addition.

members of the set

Given $\vec{u}, \vec{v} \in \mathbb{R}^n$, we define

$$\vec{u} + \vec{v} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{pmatrix}$$

we are <u>defining</u> this

- Scalar Multiplication.

Given vector $\vec{u} \in \mathbb{R}^n$ & "scalar" $a \in \mathbb{R}$ we define a vector $a\vec{u}$ by

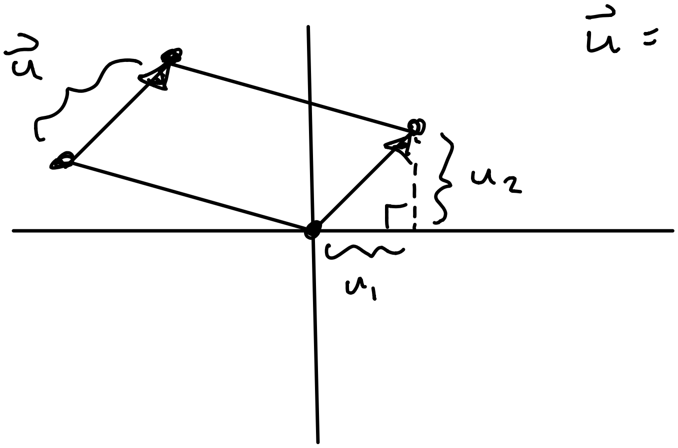
$$a\vec{u} = \begin{pmatrix} au_1 \\ au_2 \\ \vdots \\ au_n \end{pmatrix}.$$

One can check that these operations satisfy the following obvious rules:

$$\left.\begin{array}{l}
\bullet\ \vec{u} + \vec{v} = \vec{v} + \vec{u} \\
\bullet\ \vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w} \\
\bullet\ \vec{u} + \vec{0} = \vec{u} \\
\bullet\ 0\vec{u} = \vec{0} \\
\bullet\ 1\vec{u} = \vec{u} \\
\bullet\ a(b\vec{u}) = (ab)\vec{u} \\
\bullet\ a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v} \\
\bullet\ (a+b)\vec{u} = a\vec{u} + b\vec{u}
\end{array}\right\} \text{This is it.}$$

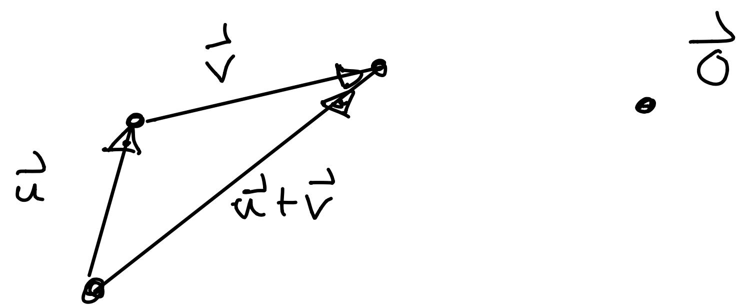
But What Does this <u>Mean</u>?

Picture:

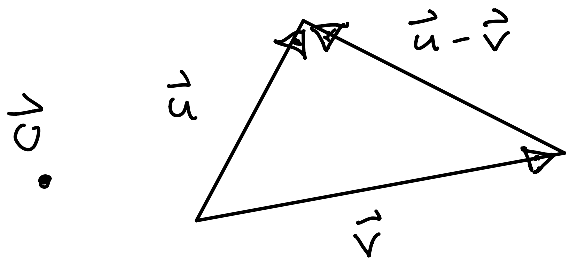


$$\vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

Addition: Head-to-Tail



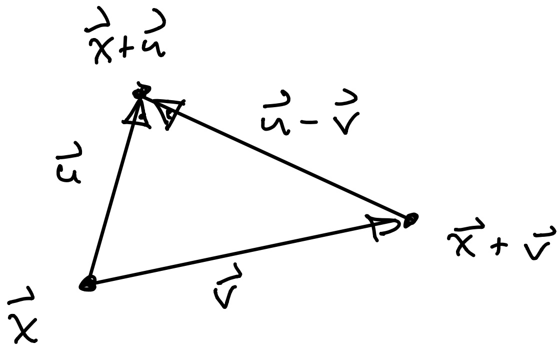
Subtraction:



$$\vec{v} + (\vec{u} - \vec{v}) = \vec{u} \checkmark$$

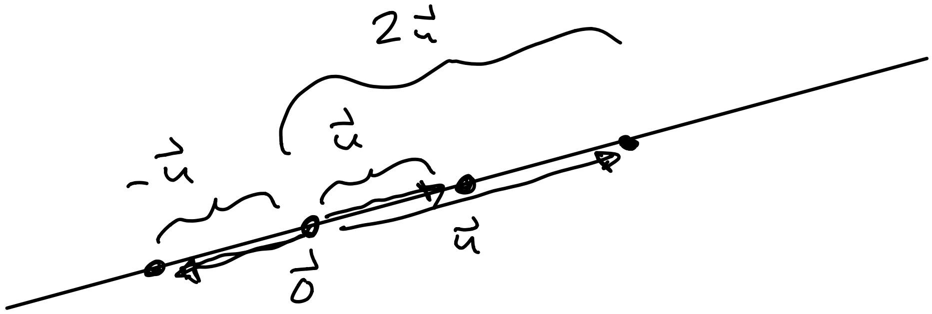
Mnemonic:

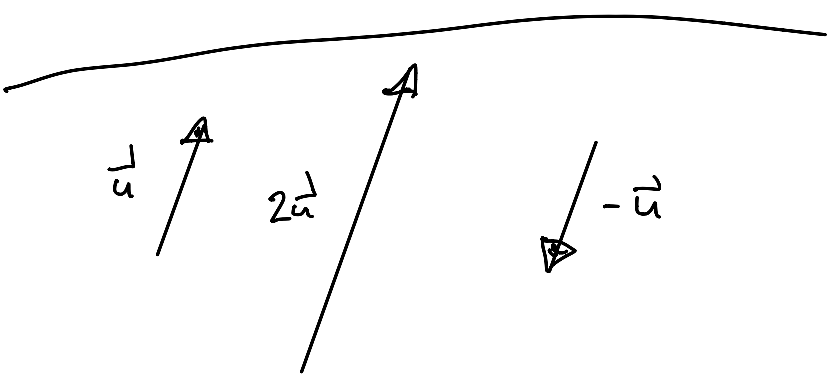
vector = its head − its tail



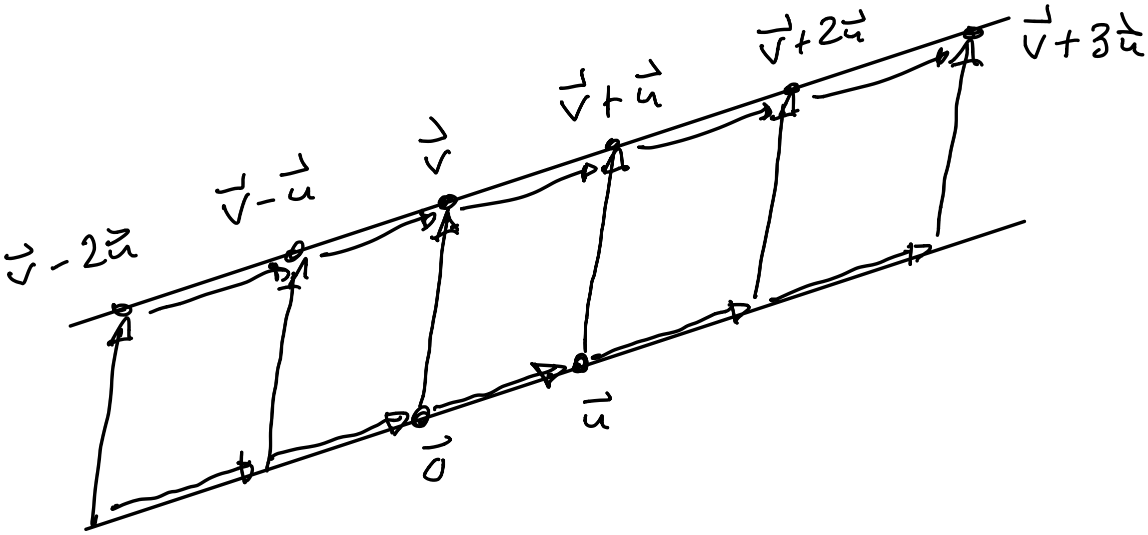
$$\vec{u}-\vec{v} = (\vec{x}+\vec{u}) - (\vec{x}+\vec{v})$$ ✓

Scalar Multiplication.





not the same $\vec{u}$.

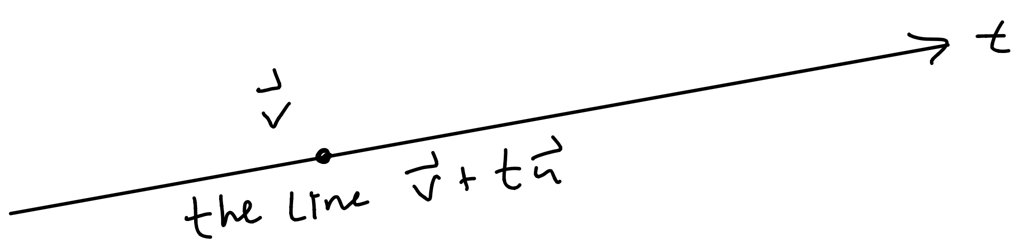


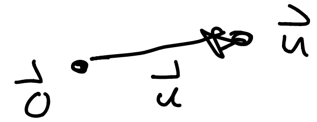
The set of points $\{\vec{v}+t\vec{u} : t \in \mathbb{R}\}$ is the <u>line</u> that is

- parallel to the vector $\vec{u}$
- contains the point $\vec{v}$.

Notice that this works in <u>any number of dimensions</u>.

"Parametrized Line"

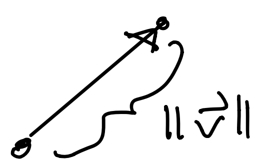


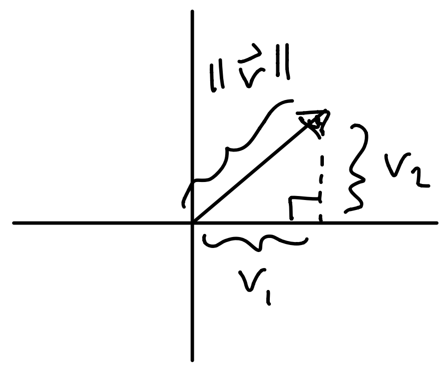


we have described vectors in terms of Cartesian "rectilinear" coordinates.

What about "magnitude & direction"?

Let $\|\vec{v}\|$ be the length of vector $\vec{v}$.



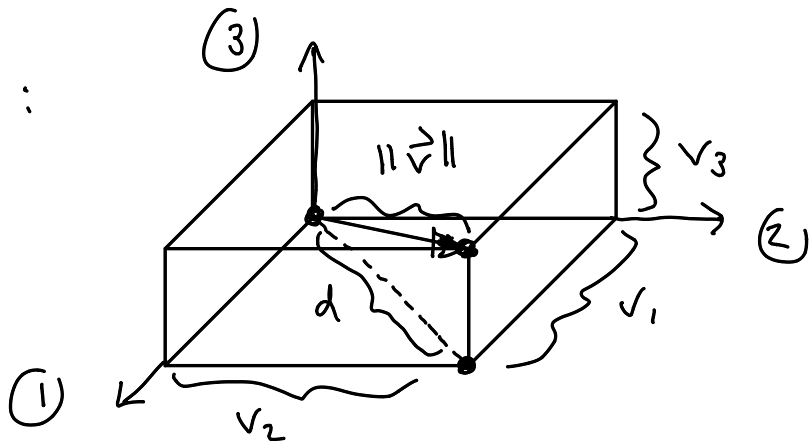


$$\|\vec{v}\|^2 = v_1^2 + v_2^2$$

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2}$$

Pythagorean Theorem.

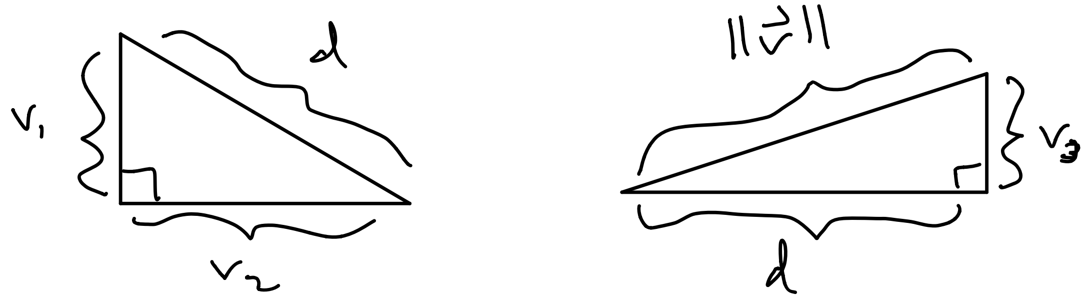
3D:



How to compute $\|\vec{v}\|$?

Guess : $\|\vec{v}\|^2 = v_1^2 + v_2^2 + v_3^2$

Proof : Two right triangles



$d^2 = v_1^2 + v_2^2$ $\qquad$ $\|\vec{v}\| = d^2 + v_3^2$

$$\|\vec{v}\|^2 = v_1^2 + v_2^2 + v_3^2$$

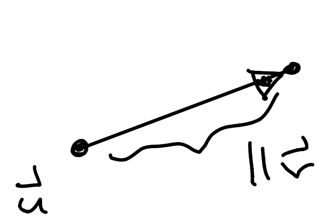
Higher Dimensions ?

Let's just say that

$$\|\vec{v}\|^2 = v_1^2 + v_2^2 + v_3^2 + \cdots + v_n^2$$

in general. Okay ?

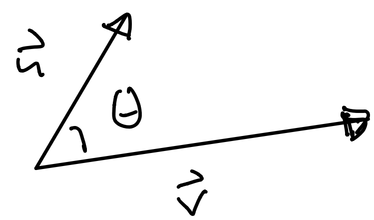
Distance:



$\|\vec{v}-\vec{u}\|$ = distance between $\vec{v}$ & $\vec{u}$.

$$= \sqrt{(v_1-u_1)^2+(v_2-u_2)^2+\cdots+(v_n-u_n)^2}$$

What About Angles / Direction ?



Let θ be angle between vectors $\vec{u}$ & $\vec{v}$.

How to compute θ ?

The answer is surprising.

There is a secret third operation of vector arithmetic.

Definition of Dot Product:
Given $\vec{u}$ & $\vec{v} \in \mathbb{R}^n$, define

$$\vec{u} \bullet \vec{v} = u_1v_1 + u_2v_2 + u_3v_3 + \cdots + u_nv_n.$$

This is a scalar, not a vector.

Algebraically, the dot product behaves like multiplication:

- $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
- $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$
- $a(\vec{u} \cdot \vec{v}) = (a\vec{u}) \cdot \vec{v} = \vec{u} \cdot (a\vec{v})$.

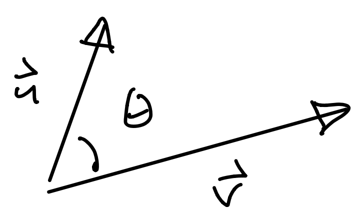
But what does this mean geometrically?

① Lengths.

$$\begin{aligned} \vec{u} \cdot \vec{u} &= u_1 u_1 + u_2 u_2 + \cdots + u_n u_n \\ &= u_1^2 + u_2^2 + \cdots + u_n^2 \\ &= \|\vec{u}\|^2 . \end{aligned}$$

$$\|\vec{u}\| = \sqrt{\vec{u} \cdot \vec{u}}$$

② Angles.



<u>I claim that</u> $\cos\theta = \dfrac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \, \|\vec{v}\|}$.

And it follows that

$\vec{u} \perp \vec{v} \iff \cos\theta = 0$

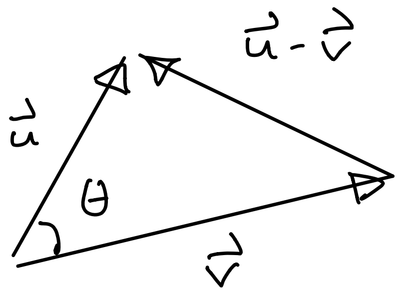
"perpendicular"

$\iff \vec{u} \cdot \vec{v} = 0$

Let me emphasize:

This gives a very easy way to check if two vectors (in any dimensional space) are perpendicular. :)

Proof: Consider the triangle:



On one hand:

$$\textcircled{*} \quad \|\vec{u}-\vec{v}\|^2 = (\vec{u}-\vec{v}) \cdot (\vec{u}-\vec{v})$$

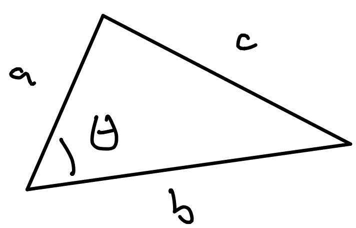
$$= \vec{u}\cdot\vec{u} - \vec{v}\cdot\vec{u} - \vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{v}$$

$$= \vec{u} \circ \vec{u} + \vec{v} \circ \vec{v} - 2\,\vec{u} \circ \vec{v}$$

$$= \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\,\vec{u} \circ \vec{v}$$

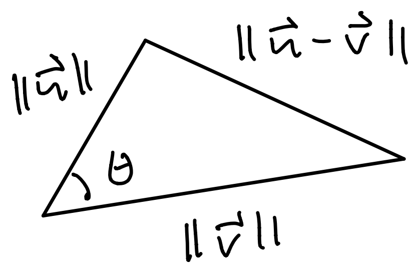
from the basic rules of arithmetic.

On the other hand, the "law of cosines" tells us that



$$c^2 = a^2 + b^2 - 2ab\cos\theta.$$

Special case: If $\theta = 90^\circ$ then this is just the Pythagorean Theorem.



(**)

$$\|\vec{u} - \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\,\|\vec{v}\|\cos\theta.$$

Finally, comparing (*) & (**),

$$\|\vec{u}-\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\,\vec{u}\circ\vec{v}$$

$$\|\vec{u}-\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\|\vec{v}\|\cos\theta$$

$$\Longrightarrow \quad \vec{u}\circ\vec{v} = \|\vec{u}\|\|\vec{v}\|\cos\theta$$

$$\Longrightarrow \quad \cos\theta = \frac{\vec{u}\circ\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|}$$

$$= \frac{\vec{u}\circ\vec{v}}{\sqrt{\vec{u}\circ\vec{u}}\,\sqrt{\vec{v}\circ\vec{v}}}$$

Thus, angles between vectors are completely determined by the dot product.

Remark: In 20th century, mathematicians discovered some exotic examples if "abstract vector spaces" with three operations

- "vector addition"
- "scalar multiplication"
- "dot product"

Example: Let S be the sample space of a random experiment. Let $X : S \to \mathbb{R}$ be any real valued function (called a random variable). The collection of random variables on S is an abstract vector space with "dot product" defined by "covariance"

$$\mathrm{Cov}(X, Y)$$

The "angle" between random variables?

$$\frac{\mathrm{Cov}(X, Y)}{\sqrt{\mathrm{Cov}(X, X)}\,\sqrt{\mathrm{Cov}(Y, Y)}}$$ is called the "correlation"

This explains <u>why</u> the correlation is a number between -1 & 1, i.e., because it is analogous to the "cosine of an angle."