HW 4 due Friday before class.

This week: Matrix Arithmetic.

Last time: If A is $m \times n$ matrix, then the function $f: \mathbb{R}^n \to \mathbb{R}^m$ defined by $f(\vec{x}) = A\vec{x}$ is a so-called "linear function":

$$(*) \begin{cases} \bullet\ f(\vec{x}+\vec{y}) = f(\vec{x}) + f(\vec{y}) & \text{preserves addition of vectors} \\ \bullet\ f(t\vec{x}) = t f(\vec{x}) & \text{preserves scalar multiplication} \end{cases}$$

[Preserves vector space structure.]

Geometrically, the function preserves lines & parallelism, i.e., sends lines to lines, sends parallel lines to parallel lines.

Conversely, given <u>any</u> function $f: \mathbb{R}^n \to \mathbb{R}^m$ satisfying properties $(*)$, there exists a unique $m \times n$ matrix A such that

$$f(\vec{x}) = A\vec{x} \quad \text{for all } \vec{x} \in \mathbb{R}^n.$$

Slogan:

Linear Functions $\mathbb{R}^n \to \mathbb{R}^m$ = $m \times n$ Matrices.

<u>Proof</u>: Given linear function $f: \mathbb{R}^n \to \mathbb{R}^m$ we define $m \times n$ matrix

$$A = \begin{pmatrix} | & | & & | \\ f(\vec{e}_1) & f(\vec{e}_2) & \cdots & f(\vec{e}_n) \\ | & | & & | \end{pmatrix}$$

where $\vec{e}_1, \ldots, \vec{e}_n \in \mathbb{R}^n$ is the "standard basis."

Then for all
$$\begin{aligned} \vec{x} &= (x_1, x_2, \ldots, x_n) \\ &= x_1\vec{e}_1 + x_2\vec{e}_2 + \cdots + x_n\vec{e}_n \end{aligned}$$

we see that

$$\begin{aligned} A\vec{x} &= x_1 f(\vec{e}_1) + \cdots + x_n f(\vec{e}_n) \\ &= f(x_1\vec{e}_1 + \cdots + x_n\vec{e}_n) \\ &= f(\vec{x}) \end{aligned}$$

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Some 2×2 Examples:

Let $f:\mathbb{R}^2\to\mathbb{R}^2$ be the "identity function" $f\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x\\y\end{pmatrix}$. What is the corresponding matrix? Since

$$f\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix} \quad \& \quad f\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix}$$

we conclude that

$$f\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$$

This matrix has a special name:

$$I=\begin{pmatrix}1&0\\0&1\end{pmatrix}.$$

"The 2×2 identity matrix"

[Remark: The identity function $\mathbb{R}^n\to\mathbb{R}^n$ corresponds to the "$n\times n$ identity matrix"

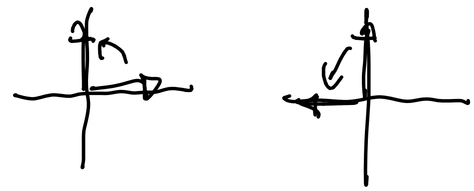
$$I=\begin{pmatrix}1&&0\\&1&\\0&&\ddots\;1\end{pmatrix}.$$

This matrix behaves like "the number 1":

$$I\vec{x} = \vec{x} \text{ for all } \vec{x}.$$

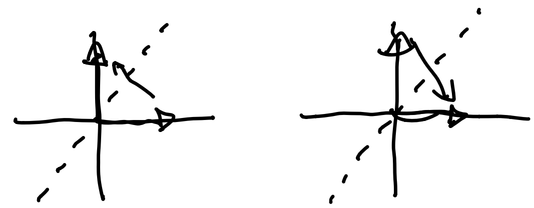
Let R be rotation c.c.w. by $90°$:

$$R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$



Let F be the reflection across $x=y$:

$$F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

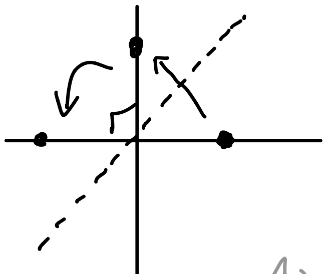


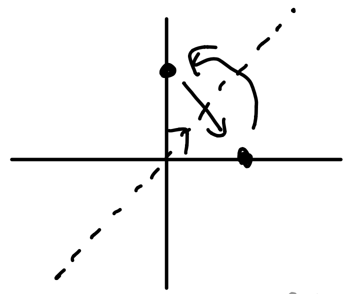
How should we define the matrix products RF & FR?

We require that $(RF)\vec{x} = R(F\vec{x})$ for all $\vec{x} \in \mathbb{R}^2$, i.e., the linear function RF is the composite function that

- <u>first</u> reflects across $x=y$,
- <u>then</u> rotates c.c.w. by $90°$.

What does this function do to the standard basis?

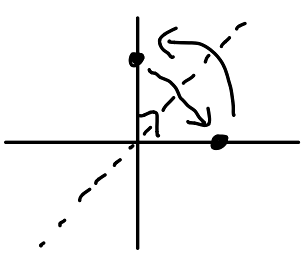


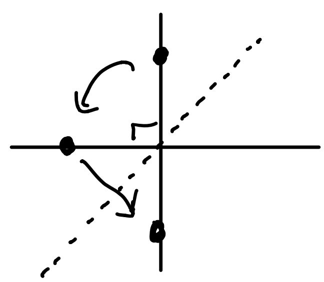


$RF\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}-1\\0\end{pmatrix}$ $\qquad$ $RF\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}$

$$\Rightarrow \quad RF = \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}.$$

Next compute FR:





$FR\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$ $\qquad$ $FR\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\-1\end{pmatrix}$

$$\Rightarrow \quad FR = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

Remarks:

- $RF \neq FR$. Matrix multiplication is not generally commutative. Reason: Composition of functions is not generally commutative.

- Easier Descriptions

$$RF\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ y \end{pmatrix}$$

"reflection across y-axis"

$$FR\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix}$$

"reflection across x-axis"

These are geometric facts that are not easy to see in terms of geometry. The <u>algebra</u> makes the geometry easier to understand!

Slogan:

Algebra is smarter than Geometry.

Example continued:

$F^2 = FF = ?$

First reflect across $x = y$ and then do it again, i.e., do nothing!

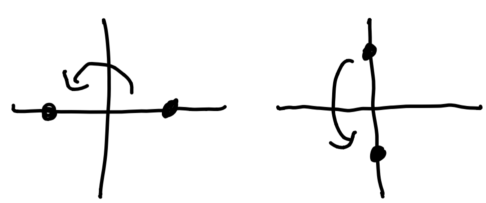
$$F^2 = I$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

$R^2 = RR = ?$

First rotate c.c.w. by 90° and then do it again, i.e., rotate c.c.w. by 180°.

$$R^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$



$$R^2 = -I$$

$$\begin{aligned} R^3 = R^2 R &= (-I)R \\ &= -IR = -R \\ &= -\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \end{aligned}$$

Meaning:

$$R^3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \text{rotate c.c.w. by } 270^\circ$$

$$= \text{rotate } \underline{\text{clockwise}} \text{ by } 90^\circ.$$

Finally:

$$R^4 = ?$$

$$= I \quad \text{do nothing.}$$

How to multiply arbitrary 2×2 matrices.

<u>Theorem</u>:

$$\underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}}_{B} = \begin{pmatrix} aa' + bc' & ab' + db' \\ ca' + dc' & cb' + dd' \end{pmatrix}.$$

i.e.

$$(ij \text{ entry of } AB) = (i\text{th row } A) \cdot (j\text{th col } B)$$

dot product.

Proof: 1st column of AB is

$$(AB)\begin{pmatrix}1\\0\end{pmatrix} = A\left(B\begin{pmatrix}1\\0\end{pmatrix}\right)$$
$$= A(\text{1st col } B)$$
$$= \begin{pmatrix}a & b\\c & d\end{pmatrix}\begin{pmatrix}a'\\c'\end{pmatrix}$$
$$= \begin{pmatrix}aa' + bc'\\ca' + dc'\end{pmatrix} \quad \checkmark$$

2nd column of AB is

$$(AB)\begin{pmatrix}0\\1\end{pmatrix} = A\left(B\begin{pmatrix}0\\1\end{pmatrix}\right)$$
$$= A(\text{2nd col } B)$$
$$= \begin{pmatrix}a & b\\c & d\end{pmatrix}\begin{pmatrix}b'\\d'\end{pmatrix}$$
$$= \begin{pmatrix}ab' + bd'\\cb' + dd'\end{pmatrix} \quad \checkmark$$

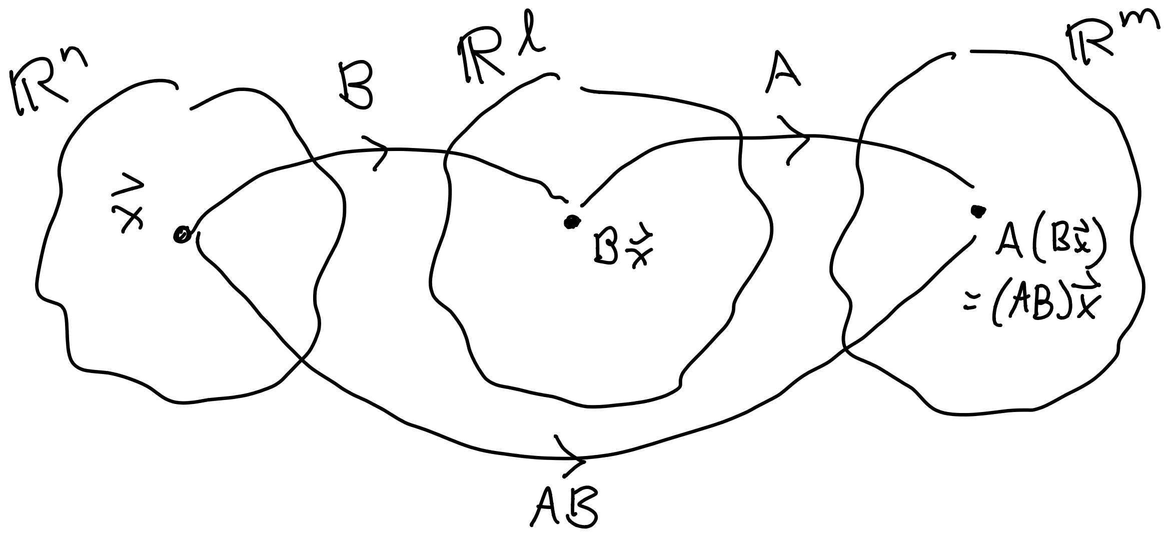
This allows us to compute with arbitrary 2×2 matrices, even if we don't know what the geometry looks like!

Apply these ideas to multiply matrices of arbitrary shapes.

Let A be $m \times l$ matrix

B be $l \times n$ matrix.

Think of these as functions:



To go in one step instead of 2 we apply the linear function

AB = "First do B, then do A"

To compute the matrix AB, consider what it does to the basis vectors
$\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n \in \mathbb{R}^n$.

$$(j\text{th column of } AB) = (AB)\vec{e}_j$$
$$= A(B\vec{e}_j)$$
$$= \underbrace{A(j\text{th column of } B)}_{\text{we know how this is defined } \checkmark}.$$

Going Further:

$$(ij \text{ entry of } AB)$$
$$= (i\text{th entry of } j\text{th column } AB)$$
$$= i\text{th entry of } A(j\text{th col } B)$$
$$= (i\text{th row } A) \cdot (j\text{th col } B)$$

dot product.

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Emphasize Shapes:

If A is $m \times \ell$ & B is $\ell \times n$ then the product AB is <u>defined</u> and has shape $m \times n$:

$$(m \times \cancel{\ell}) \cancel{\times} (\cancel{\ell} \times n) = m \times n$$

If (# columns A) $\neq$ (# rows B) then product AB is NOT DEFINED.

Example: $A = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}$ & $B = \begin{pmatrix} 1 & -1 \\ 0 & 1 \\ 2 & 0 \end{pmatrix}$

AA NOT DEFINED.

$$\begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix} =$$

Similarly, BB is not defined.

Compute AB & BA:

$$AB = \underset{2\times 3}{\begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}} \underset{3\times 2}{\begin{pmatrix} 1 & -1 \\ 0 & 1 \\ 2 & 0 \end{pmatrix}} = \underset{2\times 2}{\begin{pmatrix} 5 & -1 \\ 3 & 0 \end{pmatrix}}$$

$$BA = \underset{3\times 2}{\begin{pmatrix} 1 & -1 \\ 0 & 1 \\ 2 & 0 \end{pmatrix}} \underset{2\times 3}{\begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}} = \underset{3\times 3}{\begin{pmatrix} 0 & -1 & 1 \\ 1 & 1 & 1 \\ 2 & 0 & 4 \end{pmatrix}}$$

To end today's lecture: How does matrix transpose interact with multiplication?

Let's say

$$\begin{matrix} A \text{ is } m\times l \\ B \text{ is } l\times n \end{matrix} \longrightarrow \begin{matrix} A^T \text{ is } l\times m \\ B^T \text{ is } n\times l. \end{matrix}$$

Observe that matrices AB & $B^T A^T$ are defined with following shapes:

$$\underset{m\times k \;\; k\times n}{A\,B} \text{ is } m \times n$$

$$\underset{n\times k \;\; k\times m}{B^T A^T} \text{ is } n \times m$$

How are these related?

Theorem: $(AB)^T = B^T A^T$.

[Lemma: $\vec{x}^T \vec{y} = \vec{x} \cdot \vec{y} = \vec{y} \cdot \vec{x} = \vec{y}^T \vec{x}$.]

Proof:

ij entry of $(AB)^T$

$= ji$ entry of AB

$=$ (jth row A) (ith col B)

$=$ (ith row B^T) (jth col A^T) Lemma!

$= ij$ entry of $B^T A^T$