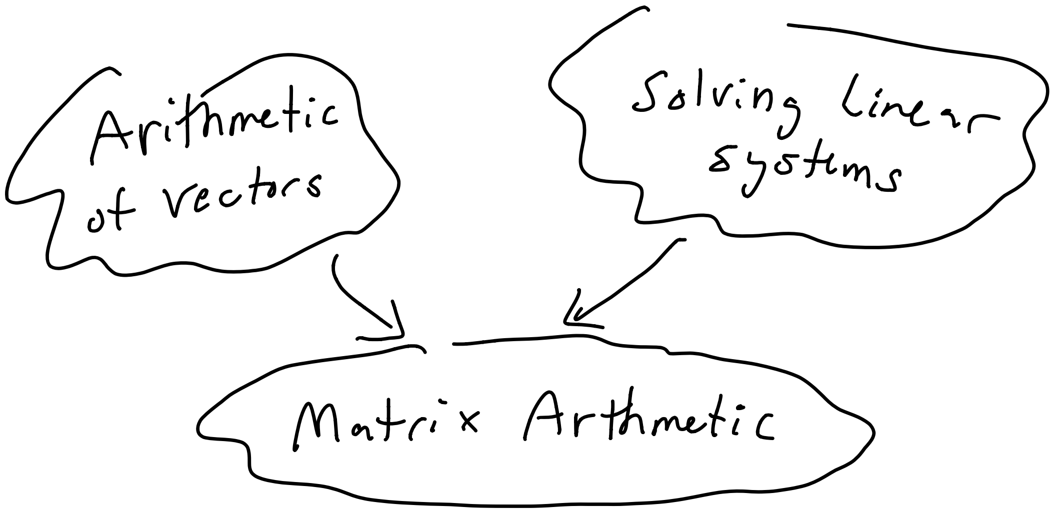
HW4 will be up before Tuesday's class, due before Friday's class.

This week: The arithmetic of matrices.



Goal is to turn all of linear algebra into mechanical computations.

Recall: Let A be $m \times n$ matrix,

$$A = \begin{pmatrix} | & & | \\ \vec{c}_1 & \cdots & \vec{c}_n \\ | & & | \end{pmatrix} = \begin{pmatrix} \text{---} \ \vec{r}_1^{\,T} \ \text{---} \\ \vdots \\ \text{---} \ \vec{r}_m^{\,T} \ \text{---} \end{pmatrix}$$

where $\vec{c}_1, \ldots, \vec{c}_n \in \mathbb{R}^m$, $\vec{r}_1, \ldots, \vec{r}_m \in \mathbb{R}^n$.

Then for any vector $\vec{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$ we define vector $A\vec{x} \in \mathbb{R}^m$ in two equivalent ways:

By columns:

$$A\vec{x} = x_1 \vec{c}_1 + \cdots + x_n \vec{c}_n$$

linear combination of columns.

By rows:

$$A\vec{x} = \begin{pmatrix} \vec{r}_1 \circ \vec{x} \\ \vdots \\ \vec{r}_m \circ \vec{x} \end{pmatrix}$$ dot products with the rows

Notation:

A^T = "A transpose"

is the $n \times m$ matrix obtained by switching rows & columns.

$$A^T = \begin{pmatrix} | & & | \\ \vec{r}_1 & \cdots & \vec{r}_m \\ | & & | \end{pmatrix} = \begin{pmatrix} \text{---} \; \vec{c}_1^{\,T} \; \text{---} \\ \vdots \\ \text{---} \; \vec{c}_n^{\,T} \; \text{---} \end{pmatrix}.$$

Special Case: Given $\vec{x}, \vec{y} \in \mathbb{R}^n$,

$$\vec{x}^T \vec{y} = (x_1 \cdots x_n) \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

$$= x_1 y_1 + \cdots + x_n y_n = \vec{x} \cdot \vec{y}.$$

(row)(column) = dot product

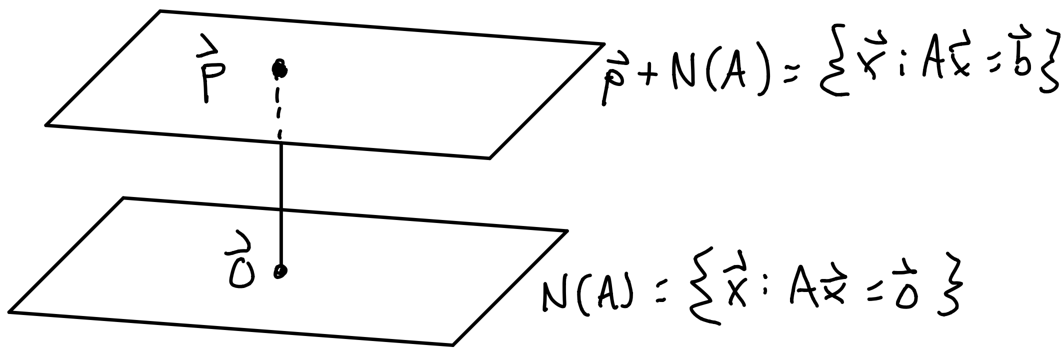
Why do we do this?

Gives a shorthand notation for systems of linear equations:

$$A\vec{x} = \vec{b}$$

System of m linear equations in the n unknowns $\vec{x} = (x_1, x_2, \cdots, x_n)$.

Picture: Solution of $A\vec{x}=\vec{b}$ is $(n-r)$-dimensional plane parallel to the nullspace $N(A)=\{\vec{x} : A\vec{x}=\vec{0}\}$:



Question: What is the most natural point $\vec{p}$ to choose?

Answer: The "orthogonal projection." We'll discuss this next week.

This week: Multiplying matrices.

How should we define the product of matrices "AB"?

Short Answer: We should define the matrix AB so that the following equation is alway true:

$$A(B\vec{x}) = (AB)\vec{x}$$

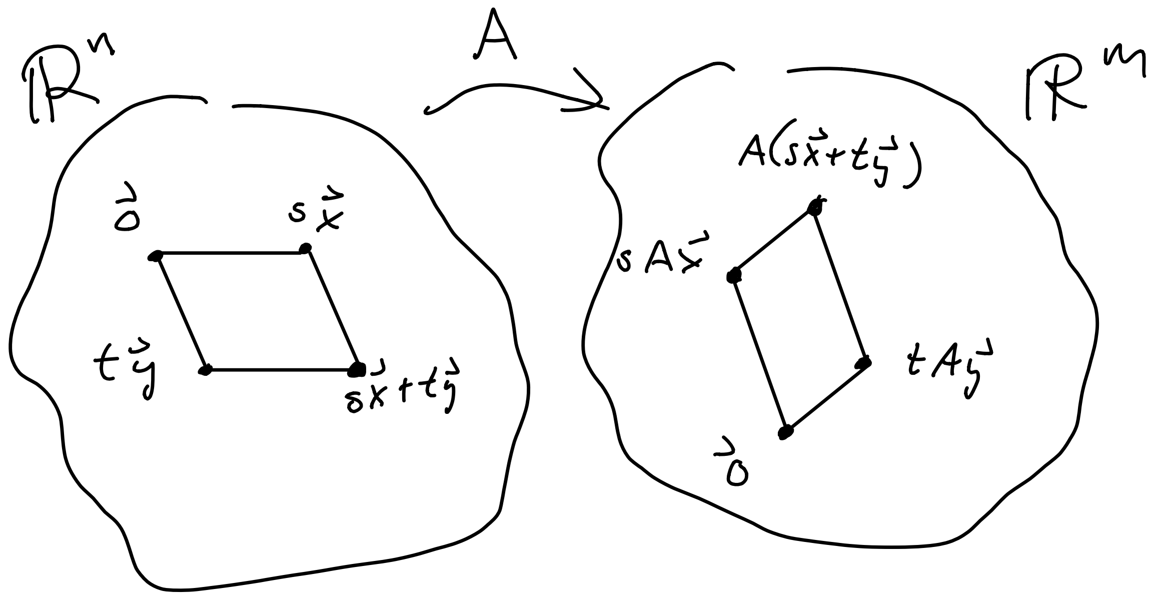
OK, but how can we actually <u>compute</u> the matrix AB?

Longer Answer: The key is the idea of a "linear function." On HW3 we saw that the function $\vec{x} \mapsto A\vec{x}$ satisfies the following property:

$$A(s\vec{x}+t\vec{y}) = s(A\vec{x}) + t(A\vec{y})$$

for all vectors $\vec{x}, \vec{y}$ & scalars s, t.

If A is $m \times n$ (m rows, n columns) then it takes each vector $\vec{x}$ in $\mathbb{R}^n$ to the vector $A\vec{x}$ in $\mathbb{R}^m$. Furthermore, this function "preserves parallelograms"



In particular, the function A sends lines to lines, which I guess is why we call it "linear."

More abstractly, let $f: \mathbb{R}^n \to \mathbb{R}^m$ be any function from $\mathbb{R}^n \to \mathbb{R}^m$. We say f is a "linear function" if

$$f(s\vec{x}+t\vec{y}) = s f(\vec{x}) + t f(\vec{y})$$

for all $\vec{x}, \vec{y} \in \mathbb{R}^n$ & $s, t \in \mathbb{R}$.

Does every linear function come from a matrix?

That is, if $f : \mathbb{R}^n \to \mathbb{R}^m$ is linear,
can we find some $m \times n$ matrix A
such that

$$f(\vec{x}) = A\vec{x} \text{ for all } \vec{x} \in \mathbb{R}^n \text{ ?}$$

Answer: YES!

Proof: Let $\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n \in \mathbb{R}^n$ be the
"standard basis vectors":

$$\left.\begin{aligned} \vec{e}_1 &= (1, 0, \dots, 0) \\ \vec{e}_2 &= (0, 1, 0, \dots, 0) \\ &\vdots \\ \vec{e}_n &= (0, \dots, 0, 1) \end{aligned}\right\} \text{definition of "Cartesian coordinates"}$$

Define $m \times n$ matrix A as follows:

$$A = \underbrace{\begin{pmatrix} f(\vec{e}_1) & f(\vec{e}_2) & \cdots & f(\vec{e}_n) \end{pmatrix}}_{n} \Big\} m$$

Then for all $\vec{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$
we have

$$\vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2 + \cdots + x_n\vec{e}_n$$
$$= (x_1, 0, \ldots, 0) + (0, x_2, 0, \ldots, 0) + \cdots + (0, \ldots, 0, x_n),$$

and hence

$$A\vec{x} = x_1 f(\vec{e}_1) + \cdots + x_n f(\vec{e}_n)$$
$$= f(x_1\vec{e}_1 + \cdots + x_n\vec{e}_n)$$
$$= f(\vec{x}) \qquad \checkmark$$

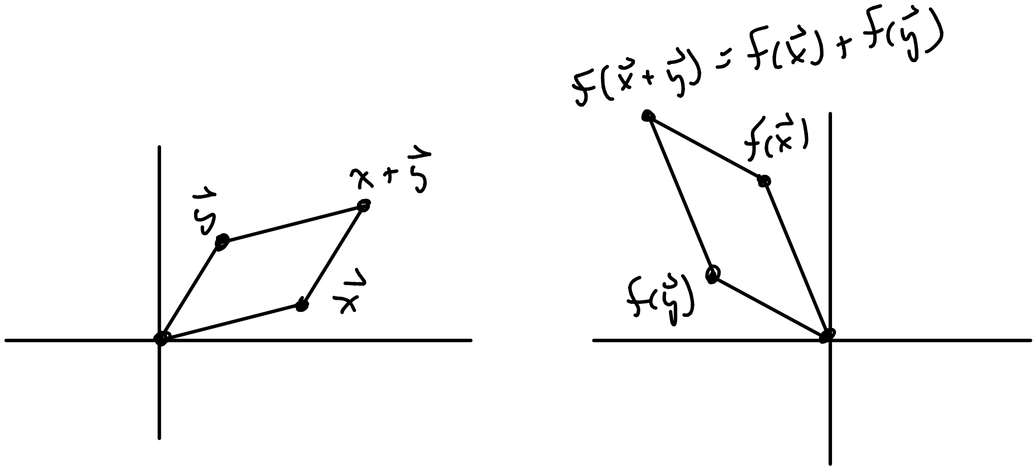
Slogan:

$$\underset{m\times n}{\text{Matrices}} = \underset{\mathbb{R}^n \to \mathbb{R}^m}{\text{Linear Functions}}.$$

Example: Some 2×2 matrices.

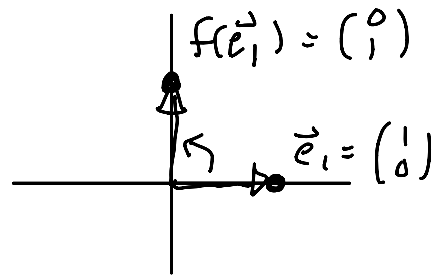
Let $f: \mathbb{R}^2 \to \mathbb{R}^2$ be the function that rotates $90°$ c.c.w. around the origin.

This function is linear:



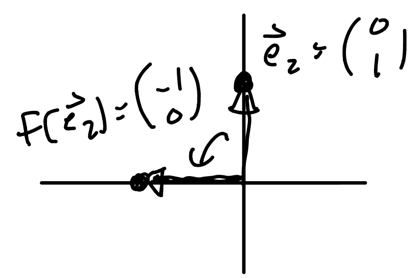
So the function has a matrix R

What is it ?



$$R = \begin{pmatrix} f(\vec{e}_1) & f(\vec{e}_2) \end{pmatrix}$$

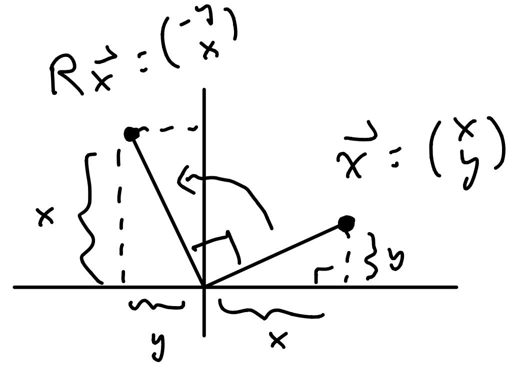
$$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$



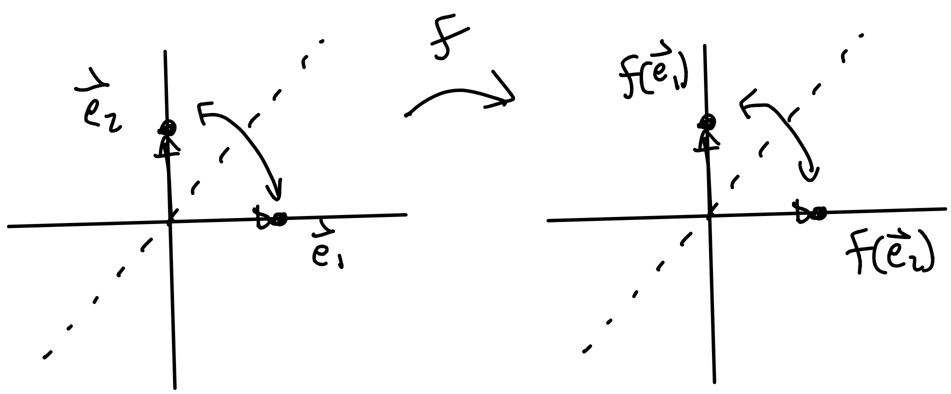
To rotate general vector $\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$

$$R\vec{x} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}$$

Picture :



<u>Next</u>, let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be reflection across the line $x = y$.



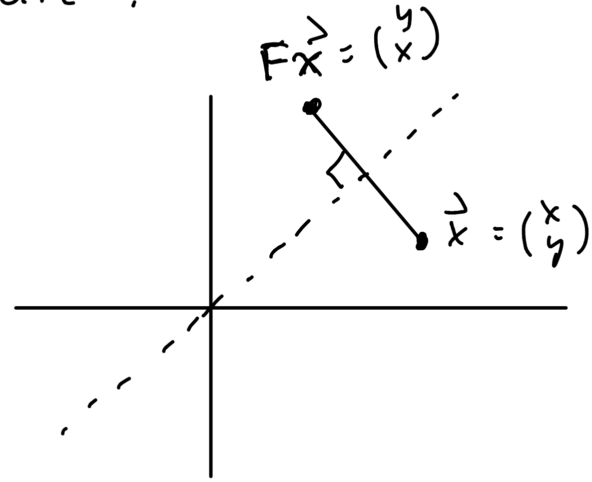
The matrix is

$$F = \begin{pmatrix} | & | \\ f(\vec{e}_1) & f(\vec{e}_2) \\ | & | \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

To reflect a general vector $\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$:

$$F\vec{x} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}.$$

Picture :



For next time : How should we define the matrices

$$FR \quad \& \quad RF \quad ?$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \ ?$$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \ ?$$