HW 3 due Friday before class.

Another example of RREF:

$$\begin{cases} x_1 + x_2 + 2x_3 + 2x_4 = 0, \\ x_1 + 2x_2 + 3x_3 + 4x_4 = 0. \end{cases} \qquad \begin{pmatrix} \vec{u}\cdot\vec{x} = 0 \\ \vec{v}\cdot\vec{x} = 0 \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 2 & 2 & 0 \\ 1 & 2 & 3 & 4 & 0 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 2 & 2 & 0 \\ 0 & 1 & 1 & 2 & 0 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & 0 \end{array}\right) \quad \text{RREF}$$

$$\begin{cases} x_1 + 0 + x_3 + 0 = 0, \\ 0 + x_2 + x_3 + 2x_4 = 0. \end{cases}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -x_3 \\ -x_3 - 2x_4 \\ x_3 \\ x_4 \end{pmatrix}$$

$$= x_3 \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ -2 \\ 0 \\ 1 \end{pmatrix}$$

a 2-plane in $\mathbb{R}^4$ passing through the origin $\vec{0}$,
i.e., a 2-dimensional <u>subspace</u> of $\mathbb{R}^4$

Geometrically?

Let $\vec{u} = (1, 1, 2, 2)$ $\quad$ $\vec{a} = (-1, -1, 1, 0)$

$\vec{v} = (1, 2, 3, 4)$ $\quad$ $\vec{b} = (0, -2, 0, 1)$

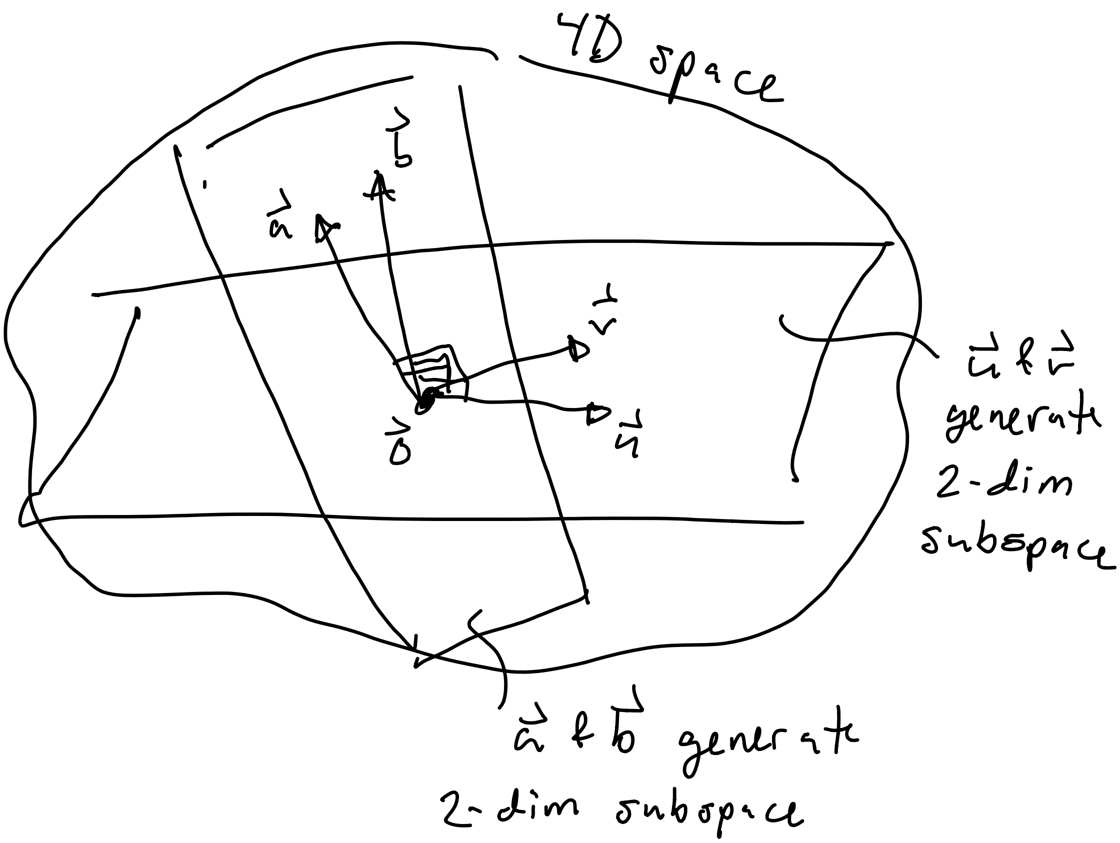
Note that

$$\vec{u} \circ \vec{a} = 0 \qquad \vec{u} \cdot \vec{b} = 0$$
$$\vec{v} \circ \vec{a} = 0 \qquad \vec{v} \circ \vec{b} = 0.$$

Two pairs of vectors,

Each pair is $\perp$ to the other pair.

What does this look like?



But what is the <u>intersection</u> of these subspaces?

I claim that these subspaces meet at a single point, namely $\vec{0}$.

Let's check. Any point $\vec{x}$ that is on both planes must satisfy

$$\vec{u}\cdot\vec{x} = 0 \qquad \vec{a}\cdot\vec{x} = 0$$
$$\vec{v}\cdot\vec{x} = 0 \qquad \vec{b}\cdot\vec{x} = 0 .$$

This gives a system of 4 linear equations in 4 unknowns:

$$\begin{cases} x_1 + x_2 + 2x_2 + 3x_4 = 0 \\ x_1 + 2x_2 + 3x_3 + 4x_4 = 0 \\ -x_1 - x_2 + x_3 + 0 = 0 \\ 0 - 2x_1 + 0 + x_4 = 0 \end{cases}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 2 & 2 & 0 \\ 1 & 2 & 3 & 4 & 0 \\ -1 & -1 & 1 & 0 & 0 \\ 0 & -2 & 0 & 1 & 0 \end{array}\right) \underset{\text{RREF}}{\rightsquigarrow} \left(\begin{array}{cccc|c} 1 & & & & 0 \\ & 1 & & & 0 \\ & & 1 & & 0 \\ & & & 1 & 0 \end{array}\right)$$

The <u>unique</u> solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Discussion: We have two 2-dimensional subspaces of $\mathbb{R}^4$

$U = \{ s\vec{u} + t\vec{v} : s, t \in \mathbb{R} \}$

$V = \{ s\vec{a} + t\vec{b} : s, t \in \mathbb{R} \}$

For all $\vec{x} \in \mathcal{U}$ & $\vec{y} \in V$ we have

$$\vec{x} \cdot \vec{y} = 0.$$

Jargon: In this case we say that $\mathcal{U}$ & V are "orthogonal complements" of each other and we write

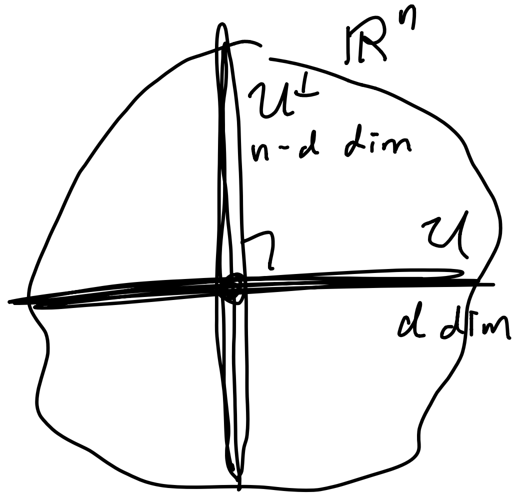
$$\mathcal{U} = V^{\perp} \quad \& \quad V = \mathcal{U}^{\perp}$$

More generally, let $\mathcal{U} \subseteq \mathbb{R}^n$ be any d-dimensional subspace of $\mathbb{R}^n$. Then we define its <u>orthogonal complement</u> as follows:

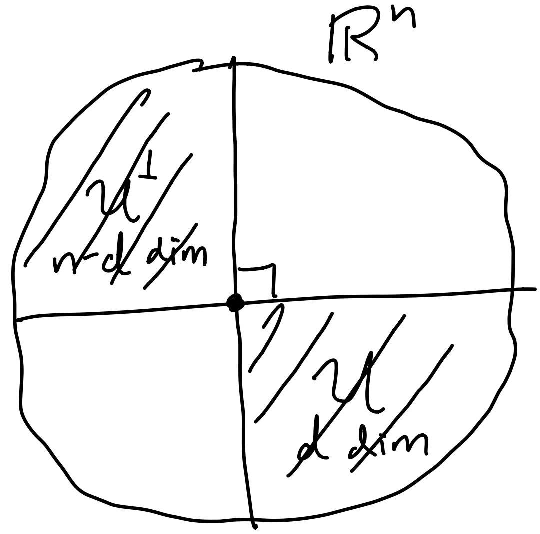
$$\mathcal{U}^{\perp} := \{ \vec{x} \in \mathbb{R}^n : \vec{x} \cdot \vec{y} = 0 \text{ for } \underline{\text{all}} \ \vec{y} \in \mathcal{U} \}.$$

Claim: $\mathcal{U}^{\perp}$ is an $(n-d)$-dimensional subspace of $\mathbb{R}^n$. [I gave an argument last time, which you will regurgitate on HW3.]

Picture (?)



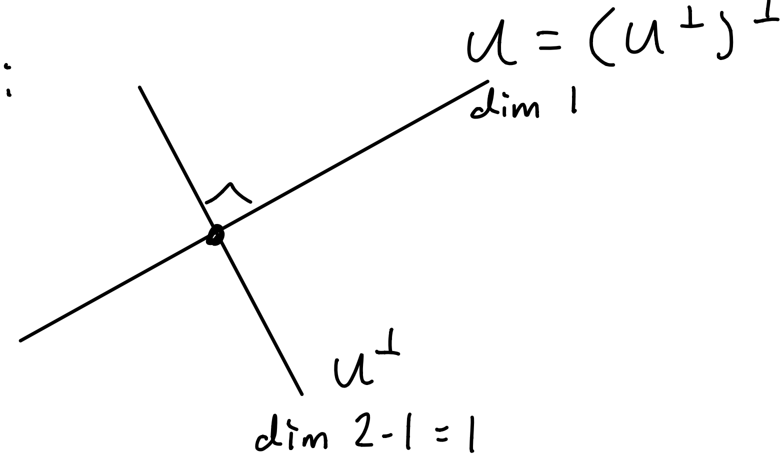
My Picture



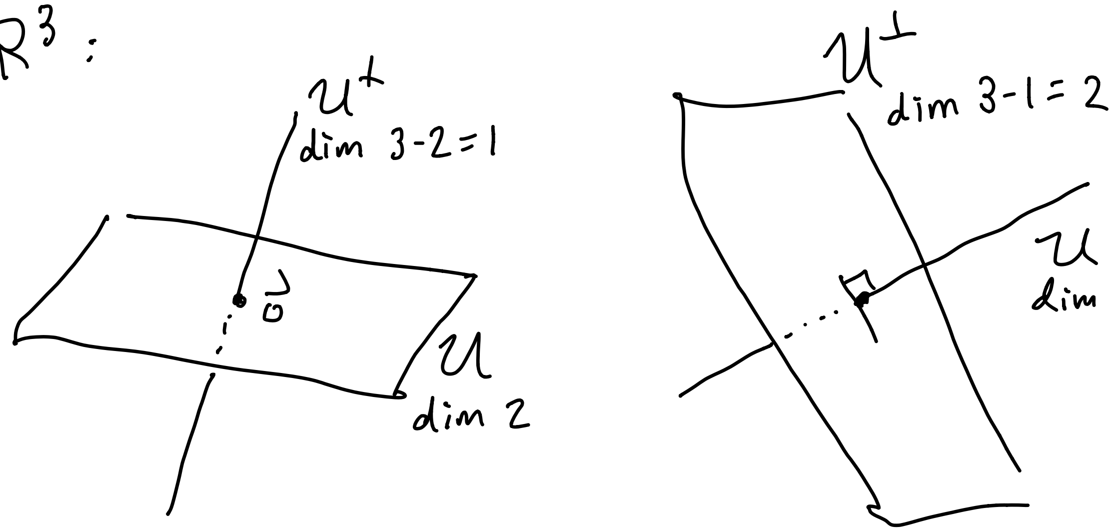
Gilbert Strang's picture

But we can only draw accurate pictures in $\mathbb{R}^2$ & $\mathbb{R}^3$.

$\mathbb{R}^2$:



$\mathbb{R}^3$:



These are the only interesting examples that we can actually draw.

Goals for this week:

- Define the notation $A\vec{x} = \vec{b}$
- State (and hopefully explain) what Strang calls the "Fundamental Theorem of Linear Algebra."

The problem of linear algebra is to solve a system of m linear equations in n unknowns:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \qquad\qquad\qquad\qquad\qquad\qquad\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

But this notation takes up too much space (and time) so we prefer to write

$$\boxed{A\vec{x} = \vec{b}}$$

What?!

Let A be the $m \times n$ matrix of coefficients.

$$A = m\left\{\overbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}^{n}\right.$$

Let $\vec{x}$ be the $n \times 1$ column of unknowns:

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$$

Let $\vec{b}$ be the $m \times 1$ column of constants

$$\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \in \mathbb{R}^m$$

Emphasize the shapes:

$$m\left\{\underbrace{\begin{pmatrix} \quad & A & \quad \end{pmatrix}}_{n}\right. \left.\underbrace{\begin{pmatrix} \vec{x} \end{pmatrix}}_{1}\right\}n = \left.\underbrace{\begin{pmatrix} \vec{b} \end{pmatrix}}_{1}\right\}m$$

$$\underbrace{(m \times \cancel{n})(\cancel{n} \times 1)}_{\text{the } n\text{'s cancel}} = (m \times 1)$$

In general, given $m \times n$ matrix A and $n \times 1$ column $\vec{x}$ we will define an $m \times 1$ column "$A\vec{x}$" as follows.

Let $A = \overbrace{\begin{pmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \\ | & | & & | \end{pmatrix}}^{n} \Big\} m$

have column vectors $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n \in \mathbb{R}^m$.

Then we define

$$A\vec{x} = A \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$= \begin{pmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \\ | & | & & | \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$:= x_1 \vec{a}_1 + x_2 \vec{a}_2 + \cdots + x_n \vec{a}_n.$$

This is maybe the most important definition in linear algebra.

Example: $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}$, $\vec{x} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

$$A\vec{x} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$:= x\begin{pmatrix} 1 \\ 1 \end{pmatrix} + y\begin{pmatrix} 1 \\ 2 \end{pmatrix} + z\begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} x + y + z \\ x + 2y + 3z \end{pmatrix}.$$

Therefore the system of two linear equations

$$\begin{cases} x + y + z = 2, \\ x + 2y + 3z = 5. \end{cases}$$

is equivalent to the single matrix equation

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

"$A\vec{x} = \vec{b}$"

Possibly Silly Case:

What if A has just one row?

$$(a_1 \ a_2 \ \cdots \ a_n)\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} := x_1 a_1 + x_2 a_2 + \cdots + x_n a_n.$$

This is just the <u>dot product</u>!

Notation: For any matrix $A = (a_{ij})$ we define the <u>transpose</u> matrix by

$$A^T = (a_{ji})$$

Example:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}^T = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix}$$

Convention: $\vec{u}$ always refers to a <u>column</u> vector. If you want to talk about a row vector, write $\vec{u}^T$

$$\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}^T = \begin{pmatrix} u_1 & u_2 & \cdots & u_n \end{pmatrix}$$

This allows us to express the dot product of column vectors in terms of matrix multiplication:

$$\vec{x} \cdot \vec{y} = \vec{x}^T \vec{y}$$

(column) $\bullet$ (column) (row)(column)

dot product matrix multiplication. (more sophisticated)

So we never have to use the notation "$\bullet$" ever again!

(unless we want to ...)

There is another way to think about the operation $A\vec{x}$ in terms of the <u>rows</u> of A.

Let
$$A = m\left\{\begin{pmatrix} \text{---}\ \vec{r}_1^{\,T}\ \text{---} \\ \text{---}\ \vec{r}_2^{\,T}\ \text{---} \\ \vdots \\ \text{---}\ \vec{r}_m^{\,T}\ \text{---} \end{pmatrix}\right.$$
$$\underbrace{\qquad\qquad}_{n}$$

where each of the m row vectors has n entries: $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_m \in \mathbb{R}^n$.

Then for all $\vec{x} \in \mathbb{R}^n$ we define

$$A\vec{x} = \begin{pmatrix} \text{---}\ \vec{r}_1^{\,T}\ \text{---} \\ \vdots \\ \text{---}\ \vec{r}_m^{\,T}\ \text{---} \end{pmatrix} \vec{x}$$

$$:= \begin{pmatrix} \vec{r}_1^{\,T}\vec{x} \\ \vec{r}_2^{\,T}\vec{x} \\ \vdots \\ \vec{r}_m^{\,T}\vec{x} \end{pmatrix} = \begin{pmatrix} \vec{r}_1 \cdot \vec{x} \\ \vec{r}_2 \cdot \vec{x} \\ \vdots \\ \vec{r}_m \cdot \vec{x} \end{pmatrix}.$$

One can check that this is equivalent to the previous definition:

$$\begin{pmatrix} \vec{r}_1^{\,T} \vec{x} \\ \vec{r}_2^{\,T} \vec{x} \\ \vdots \\ \vec{r}_m^{\,T} \vec{x} \end{pmatrix} = x_1 \vec{a}_1 + x_2 \vec{a}_2 + \cdots + x_n \vec{a}_n .$$

maybe <u>you</u> can check.

The strength of matrix notation is that it simultaneously encodes information about row vectors & column vectors.

Preview of the Fundamental Theorem.

For any matrix A

\# independent columns = \# independent rows

$$\operatorname{rank}(A) = \operatorname{rank}(A^T).$$