HW3 due Friday __before__ class.

Last time I showed you an example of Gaussian Elimination:

$$\begin{array}{l} ① \\ ② \\ ③ \end{array} \begin{cases} x_1 + 3x_2 + 3x_3 + x_4 = 5 \\ \qquad\qquad 2x_3 + 2x_4 = 3 \\ x_1 + 3x_2 + x_3 - x_4 = 2 \end{cases}$$

$$\xrightarrow[\text{symbols}]{\text{remove unnecessary}} \left(\begin{array}{cccc|c} 1 & 3 & 3 & 1 & 5 \\ 0 & 0 & 2 & 2 & 3 \\ 1 & 3 & 1 & -1 & 2 \end{array}\right)$$

$$\xrightarrow[\text{row operations)}]{\text{sequence of EROs (elementary}} \left(\begin{array}{cccc|c} 1 & 3 & 0 & -2 & 1/2 \\ 0 & 0 & 1 & 1 & 3/2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow[\text{symbols back.}]{\text{put the}} \begin{cases} x_1 + 3x_2 + 0 - 2x_4 = 1/2 \\ 0 + 0 + x_3 + x_4 = 3/2 \end{cases}$$

The resulting system (called the RREF)

is much simpler but it still has the same solutions as the original.

The original system represents the intersection of three "3-dimensional planes" living in 4-dimensional space.

The intersection is a 2-dimensional plane:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 0 \\ 3/2 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 2 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

Remark: We can't visualize this but we can do the calculation!

Next: Gaussian Elimination for Computers.

Input: A matrix (rectangular array) of numbers.

[$m \times n$ matrix has m rows & n columns].

Algorithm has 3 Steps.

Step 1:

- Move pointer to leftmost nonzero column.
- Swap rows (Type I) to obtain nonzero entry in first row of this column.
  This entry is the <u>pivot</u>.
- Elimininate all entries below the pivot. (Type III EROs).
- Delete the first row and <u>repeat</u> on the remaining rows.

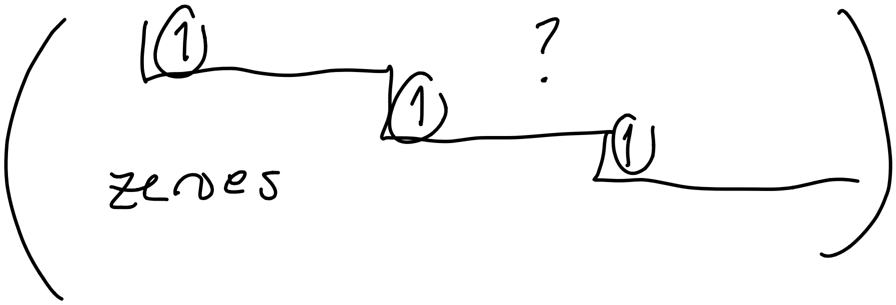
Result: "Echelon form"

$$\begin{pmatrix} 0 & * & & & & & & & & & \\ 0 & 0 & 0 & 0 & * & & & & & & \\ 0 & 0 & \cdots & & & 0 & * & & & & \\ 0 & 0 & \cdots & & & 0 & \cdots & & & & 0 \end{pmatrix}$$

"Echelon" = "Staircase"

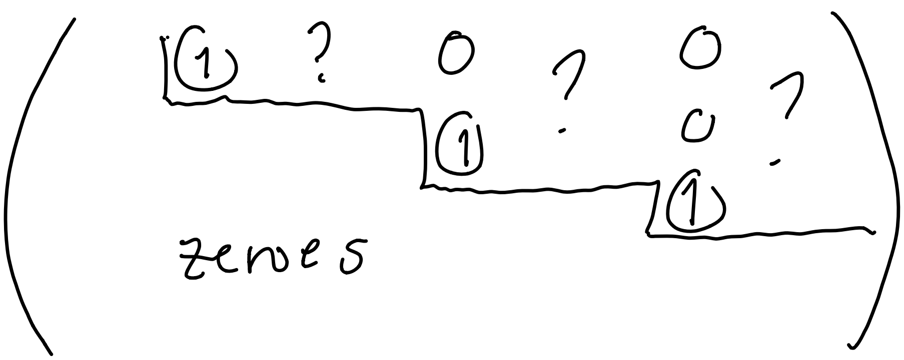
Step 2:

Scale the rows (Type II EROs) to convert pivot entries to 1s.



Step 3:

Eliminate all entries <u>above</u> the pivots (EROs of Type III).



END of ALGORITHM.

Output : A matrix of same shape in RREF (reduced row echelon form).

<u>Why</u> do we do it this way?

Theorem : The RREF is <u>unique</u>!

Meaning : No matter the sequence of EROs that you use, the resulting RREF will always be the same.

Let's do a few examples on my computer.

Shape $\left(\begin{array}{ccc|c} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{array}\right)$

$$\left(\begin{array}{ccc|c} 1 & 5 & 7 & 8 \\ 2 & 6 & 3/7 & 0 \end{array}\right)$$

$$\xrightarrow{\text{RREF}} \quad \begin{array}{c} \begin{array}{cccc} x & y & z & \end{array} \\ \left(\begin{array}{ccc|c} 1 & 0 & -279/28 & -12 \\ 0 & 1 & 95/28 & 4 \end{array}\right) \end{array}$$

Geometrically: Two planes in 3D intersecting in a line

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -12 + (279/28)z \\ 4 - (95/28)z \\ z \end{pmatrix}$$

$$= \begin{pmatrix} -12 \\ 4 \\ 0 \end{pmatrix} + z \begin{pmatrix} 279/28 \\ -95/28 \\ 1 \end{pmatrix}.$$

Shape:

$$\left(\begin{array}{ccc|c} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 2 & 2 & 4 & 0 \\ 1 & 6 & 3 & 4 \\ 0 & 1 & 2 & 2 \end{array}\right)$$

$$\xrightarrow{\text{RREF}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 2/3 \\ 0 & 0 & 1 & 2/3 \end{array}\right)$$

Guess: We expect to get a <u>unique</u> solution corresponding to 3 planes intersecting at a point.

And we did:

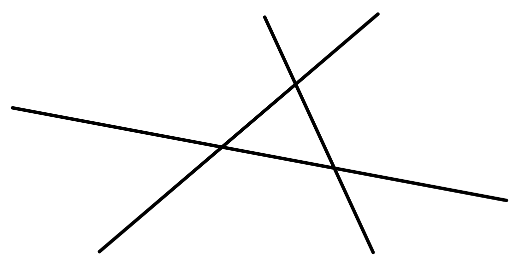
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 2/3 \\ 2/3 \end{pmatrix}.$$

Shape: $\left(\begin{array}{cc|c} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{array}\right)$

$$
\begin{array}{c} ① \\ ② \\ ③ \end{array}
\left(\begin{array}{cc|c} 7 & -2 & 3 \\ 8 & 6 & 1 \\ 16 & 12 & 2 \end{array}\right) \rightsquigarrow \; ?
$$

What do we expect?

3 lines in the plane. We expect to get NO SOLUTION:



And my computer says:

$$
\xrightarrow[\rightsquigarrow]{\text{RREF}} \left(\begin{array}{cc|c} 1 & 0 & 10/29 \\ 0 & 1 & -17/58 \\ 0 & 0 & 0 \end{array}\right)
$$

OOPS! We got a point! $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 10/29 \\ -17/58 \end{pmatrix}$

Reason: Lines ② & ③ are the same

equation because ③ = 2②.

Let's try to be more random:

$$\left(\begin{array}{cc|c} 1 & 2 & 3 \\ 2 & 1 & 0 \\ 1 & 1 & 2 \end{array}\right) \xrightarrow{\text{RREF}} \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right)$$

What does this mean?

The 3rd row of the RREF says that

$$0x + 0y = 1,$$

which is <u>impossible</u>.

Lesson:

- A row of zeroes in RREF can be ignored because it says
$$0x_1 + 0x_2 + \cdots + 0x_n = 0,$$
which gives NO INFORMATION.

- A row of the form
$$0 \quad 0 \quad \cdots \quad 0 \mid c \qquad (c \neq 0)$$

means there is NO SOLUTION because the equation

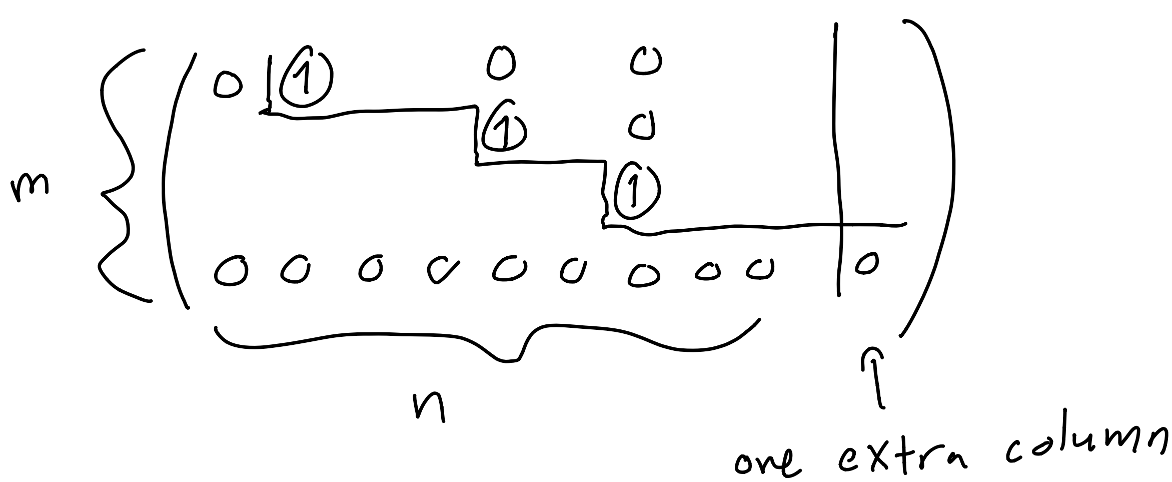
$$0x_1 + 0x_2 + \cdots + 0x_n = c \quad (\neq 0)$$

has NO SOLUTION.

We now have the tools to solve any linear system via Gaussian Elim.

What does the general situation look like?

Start with m linear equations in n unknowns. Then RREF:



- If a pivot appears in final column then there is no solution.
- The solution (if it exists) is a d-dimensional plane of the form

$$\vec{p} + t_1 \vec{u}_1 + t_2 \vec{u}_2 + \cdots + t_d \vec{u}_d,$$

where $t_1, t_2, \ldots, t_d$ are the <u>free variables</u> from RREF.

Note that

$$\begin{aligned} d &= \#\text{ free variables} \\ &= n - \#\text{ pivot variables} \leq n \checkmark \end{aligned}$$

- Jargon: The number of pivot variables is called the <u>rank</u> of the linear system.

$$r = \#\text{ pivot variables}.$$

- If we get a row of zeros, this means that there was some linear relation among the original rows. Why? Because every row of the RREF is some linear combination of the original rows.

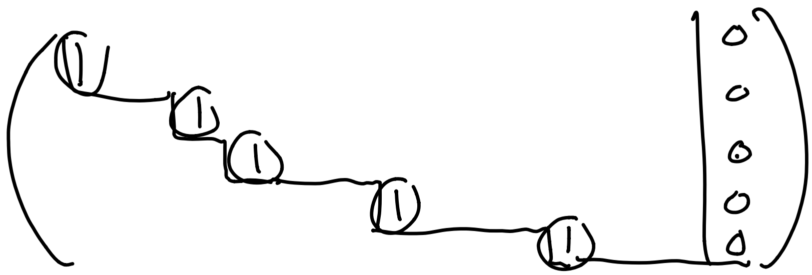
This leads to an important observation that will help you with HW3 Problem 5.

Theorem: Let $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_d \in \mathbb{R}^n$ be linearly independent vectors. Then the linear system

$$\begin{cases} \vec{u}_1 \cdot \vec{x} = 0 \\ \vec{u}_2 \cdot \vec{x} = 0 \\ \vdots \\ \vec{u}_d \cdot \vec{x} = 0 \end{cases} \qquad \left(\begin{array}{l} \#\text{equations} = d \\ \#\text{unknowns} = n \end{array} \right)$$

has a solution that is an <u>$(n-d)$-dimensional subspace</u> of $\mathbb{R}^n$.

Proof : Consider the RREF :



- rightmost column has all zeroes.
  (starting from zeros, EROs can only produce zeroes).

- Every row must have a pivot.
  Equivalently, there is no row of zeroes.

Reason: If there were a zero row we would get a nontrivial relation

$$t_1 \vec{u}_1 + t_2 \vec{u}_2 + \cdots + t_d \vec{u}_d = \vec{0}.$$

But this contradicts our assumption that the vectors $\vec{u}_1, \ldots, \vec{u}_d$ are linearly independent.

- Hence

  # pivot variables
  $= \#$ rows
  $= d$.

- Finally,

  dimension of the solutions
  $= \#$ free variables
  $= n - \#$ pivot variables
  $= n - \#$ rows
  $= n - d$.

///

I apologize that this argument was quite abstract. We will keep talking about these ideas; eventually it will make more sense to you.