HW5 is due <u>now</u>.

Quiz 5 tomorrow, beginning of class.

Final Project due Friday, <u>noon</u> (12pm):

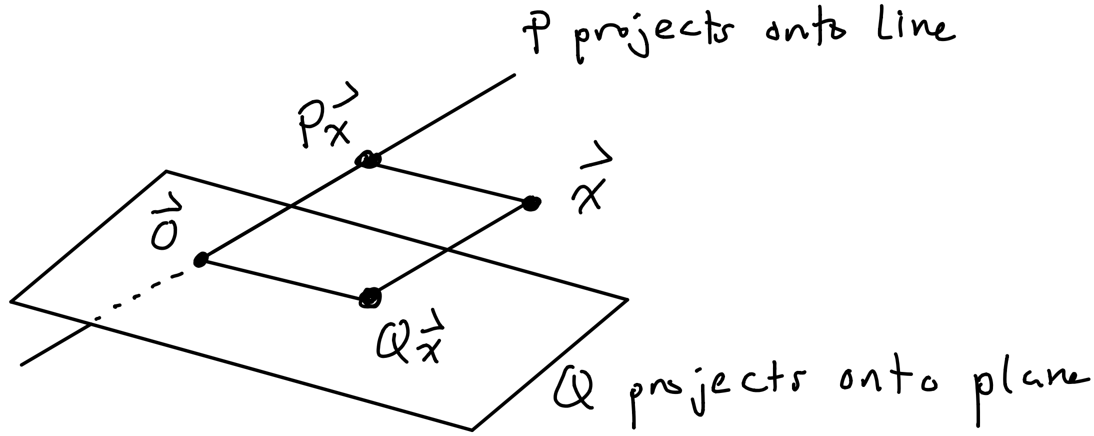
Write a summary (not an essay) of what you learned in this course.

Today: HW5 Discussion.

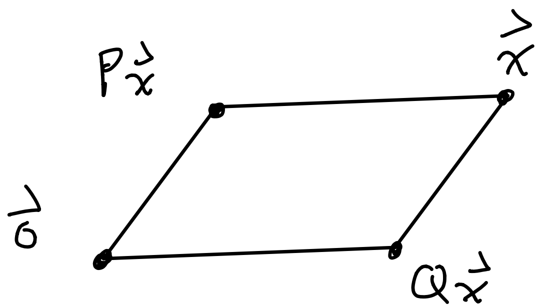
Problem 2: Jargon: A projection matrix P satisfies $P^2 = P$. We say that projections P & Q are <u>complementary</u> if

$$PQ = 0$$
$$P + Q = I.$$

Geometrically, this means that P & Q project onto "complementary subspaces," for example a line & a plane in 3D space:



These projections are not necessarily "orthogonal," i.e., they do not necessarily project at right angles. But we still have, for any point $\vec{x}$, a parallelogram



$$\vec{x} = P\vec{x} + Q\vec{x}$$
$$\vec{x} = (P+Q)\vec{x}$$

Since this holds for all $\vec{x}$ we conclude that $P + Q = I$.

Jargon: If P satisfies

$$P^2 = P \quad \underline{\text{and}} \quad P^T = P$$

then we say that P is an "orthogonal projection," i.e., it projects at right angles onto a certain subspace.

[Warning: This does not mean that P is an "orthogonal matrix":

$$P^{-1} \neq P^T.$$

In fact, any nontrivial projection is non-invertible.]

If $P^2 = P$, $P^T = P$, let's define

$$Q = I - P.$$

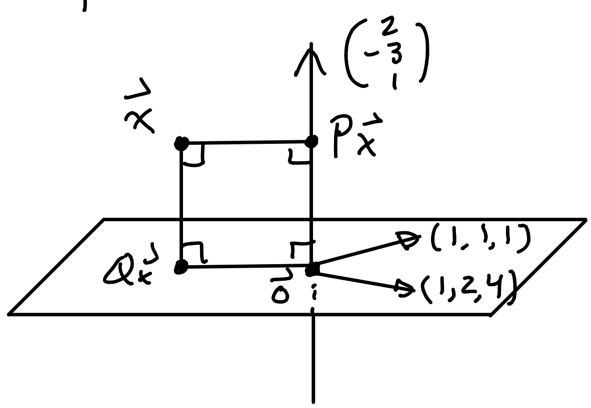
Observe that

$$\begin{aligned} Q^2 &= (I-P)(I-P) \\ &= I - 2P + P^2 \\ &= I - 2P + P = I - P = Q. \end{aligned}$$

$$\begin{aligned} Q^T &= (I-P)^T \\ &= I^T - P^T = I - P = Q, \end{aligned}$$

hence Q is also an orthogonal projection. Furthermore, P & Q are complementary ; they project onto "orthogonal complementary subspaces."

Example :



[Remark : In coordinates, a general (not necessarily orthogonal) projection matrix looks like

$$P = A(B^T A)^{-1} B^T$$

for some matrices A, B of the same shape. If $A = B$ then the projection is orthogonal.]

Problem 1: Fit a parabola $y = a + bx + cx^2$ to four data points:

$(x, y) = (1,1), (2,4), (3,3), (4,2)$.

$$\left.\begin{array}{lcl} (1,1) \rightsquigarrow & & 1 = a + b + c \\ (2,4) \rightsquigarrow & & 4 = a + 2b + 4c \\ (3,3) \rightarrow & & 3 = a + 3b + 9c \\ (4,2) \rightarrow & & 2 = a + 4b + 16c \end{array}\right\}$$

$$X \vec{a} = \vec{y}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 3 \\ 2 \end{pmatrix}$$

NO SOLUTION!

$$X^T X \vec{a} = X^T \vec{y}$$

$$\begin{pmatrix} 4 & 10 & 30 \\ 10 & 30 & 100 \\ 30 & 100 & 354 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 10 \\ 26 \\ 76 \end{pmatrix}$$

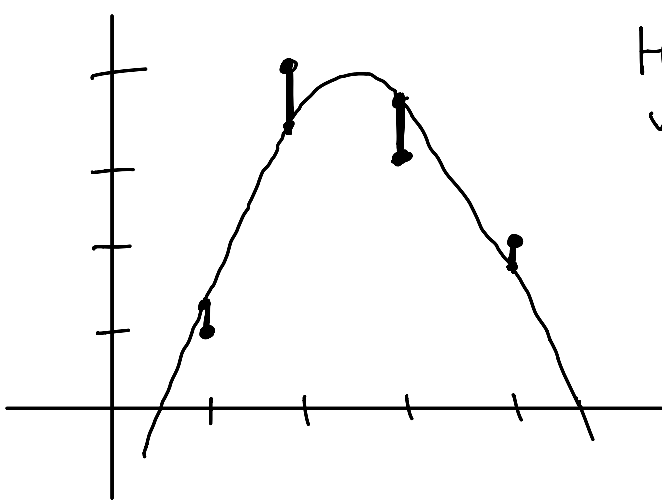
This has a unique solution:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -3 \\ 26/5 \\ -1 \end{pmatrix}$$

So the OLS best fit parabola is

$$y = -3 + \frac{26}{5}x - x^2$$

Picture:



How is this a "best fit"?

OLS: $X^T X \vec{a} = X^T \vec{y}$ minimizes

$\|X\vec{a} - \vec{y}\|^2$ = sum of squares of the <u>vertical</u> errors.

Problem 3: Write $c = \cos\theta$, $s = \sin\theta$.

$$R_\theta = \begin{pmatrix} c & -s \\ s & c \end{pmatrix}, \quad F_\theta = \begin{pmatrix} c & s \\ s & -c \end{pmatrix}, \quad P_\theta = \begin{pmatrix} c^2 & cs \\ cs & s^2 \end{pmatrix}.$$

Rotation, reFlection, Projection.
(or Flip)

$\det(R_\theta) = c^2 + s^2 = 1.$

$\det(F_\theta) = -c^2 - s^2 = -1.$

$\det(P_\theta) = c^2 s^2 - (cs)^2 = 0.$

Recall: A^{-1} exists $\Leftrightarrow \det(A) \neq 0$.

So R_θ & F_θ are invertible, in fact

$$R_\theta^{-1} = R_{-\theta} \quad (= R_\theta^T)$$

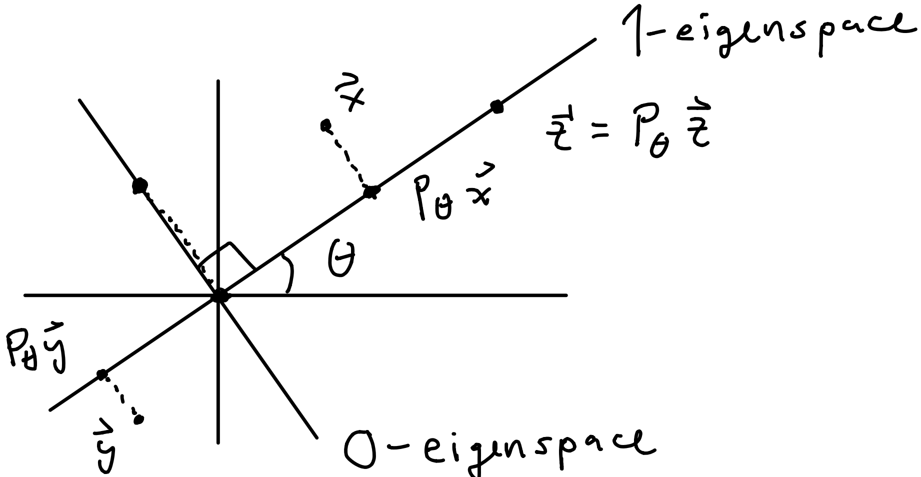
(to undo rotation by θ, rotate by $-\theta$.)

$$F_\theta^{-1} = F_\theta.$$

(to undo a reflection, do it again.)

Since $\det(P_\theta) = 0$, we see that P_θ^{-1} does not exist.

Indeed, any projection will send some nonzero vector to zero.



In fact, any vector on the line $t\begin{pmatrix}-s\\c\end{pmatrix}$ gets sent to zero.

More generally, we see that P_θ has eigenvalues 1 & 0 with

1-eigenspace = line $t\begin{pmatrix}c\\s\end{pmatrix}$

0-eigenspace = line $t\begin{pmatrix}-s\\c\end{pmatrix}$

Discussion: Let $A = \begin{pmatrix}a & b\\c & d\end{pmatrix}$.

Eigenvalues are the roots of the quadratic equation:

$$\det\begin{pmatrix} a-x & b \\ c & d-x \end{pmatrix} = 0$$

$$(a-x)(d-x) - bc = 0$$

$$x^2 - (a+d)x + (ad-bc) = 0.$$

Suppose that the eigenvalues are λ & μ so that

$$x^2 - (a+d)x + (ad-bc) = (x-\lambda)(x-\mu)$$

$$= x^2 - (\lambda+\mu)x + \lambda\mu.$$

Compare coefficients:

$$\lambda+\mu = a+d$$

$$\lambda\cdot\mu = ad-bc.$$

More generally, for square matrix A,

sum of eigenvalues $=$ trace(A) $=$ sum of diagonal entries.

product of eigenvalues $= \det(A)$.

Problems 4 & 5 : $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$.

Eigenvalues : -1 & 5.

Eigenvectors :

(-1) - eigenspace = line $t\begin{pmatrix} 1 \\ -1 \end{pmatrix}$,

5 - eigenspace = line $t\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Check :

$$\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = -1\begin{pmatrix} 1 \\ -1 \end{pmatrix} \checkmark$$

$$\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \end{pmatrix} = 5\begin{pmatrix} 1 \\ 2 \end{pmatrix} \checkmark$$

What about

$$A^n = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}^n$$

$$e^{At} = e^{\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} t}$$

$$= I + At + A^2\frac{t^2}{2} + A^3\frac{t^3}{6} + \cdots$$

What are the eigenvalues & vectors of these matrices?

A^n has eigenvalues $(-1)^n$ & 5^n.

$(-1)^n$-eigenspace = line $t\begin{pmatrix}1\\-1\end{pmatrix}$

5^n-eigenspace = line $t\begin{pmatrix}1\\2\end{pmatrix}$.

e^{At} has eigenvalues e^{-t} & e^{5t}

e^{-t}-eigenspace = line $t\begin{pmatrix}1\\-1\end{pmatrix}$

e^{5t}-eigenspace = line $t\begin{pmatrix}1\\2\end{pmatrix}$.

Use this to solve two problems.

(1) $\begin{cases} x_0 = 5 \\ y_0 = 4 \end{cases}$ $\begin{cases} x_{n+1} = x_n + 2y_n \\ y_{n+1} = 4x_n + 3y_n \end{cases}$.

Let $\vec{x}_n = \begin{pmatrix}x_n\\y_n\end{pmatrix}$.

General solution: $\vec{x}_n = A^n \vec{x}_0$.

Express

$$\vec{x}_0 = \begin{pmatrix} 5 \\ 4 \end{pmatrix} = 2\begin{pmatrix} 1 \\ -1 \end{pmatrix} + 3\begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Then

$$\begin{aligned} \vec{x}_n &= A^n \vec{x}_0 \\ &= A^n\left(2\begin{pmatrix} 1 \\ -1 \end{pmatrix} + 3\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right) \\ &= 2A^n\begin{pmatrix} 1 \\ -1 \end{pmatrix} + 3A^n\begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ &= 2(-1)^n\begin{pmatrix} 1 \\ -1 \end{pmatrix} + 3\cdot 5^n\begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} 3\cdot 5^n + 2(-1)^n \\ 6\cdot 5^n - 2(-1)^n \end{pmatrix} \end{aligned}$$

Hence $x_n = 3\cdot 5^n + 2(-1)^n$

$y_n = 6\cdot 5^n - 2(-1)^n$.

These are explicit formulas

(2) $\begin{cases} x(0) = 5 \\ y(0) = 4 \end{cases}$ $\begin{cases} x'(t) = x(t) + 2y(t) \\ y'(t) = 4x(t) + 3y(t). \end{cases}$

"System of ordinary differential equations (ODEs)"

Let $\vec{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$

= a moving point in the plane.

General Solution:

$$\vec{x}(t) = e^{At}\,\vec{x}(0)$$

$$= e^{At}\begin{pmatrix} 5 \\ 4 \end{pmatrix}$$

$$= e^{At}\left[2\begin{pmatrix} 1 \\ -1 \end{pmatrix} + 3\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right]$$

$$= 2e^{At}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + 3e^{At}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$= 2e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + 3e^{5t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 3e^{5t} + 2e^{-t} \\ 6e^{5t} - 2e^{-t} \end{pmatrix}.$$

Hence:

$$x(t) = 3e^{5t} + 2e^{-t}$$
$$y(t) = 6e^{5t} - 2e^{-t}.$$

[Compute $x'(t), y'(t)$ and check that this is correct:

$$\begin{aligned} x'(t) &= 3 \cdot 5e^{5t} + 2(-1)e^{-t} \\ &= 15e^{5t} - 2e^{-t} \\ &= (3e^{5t} + 2e^{-t}) + 2(6e^{5t} - 2e^{-t}) \\ &= x(t) + 2y(t) \quad \checkmark \end{aligned}$$

$$\begin{aligned} y'(t) &= 6 \cdot 5e^{5t} - 2(-1)e^{-t} \\ &= 4(3e^{5t} + 2e^{-t}) + 3(6e^{5t} - 2e^{-t}) \\ &= 4x(t) + 3y(t) \quad \checkmark \end{aligned}$$

It works!]

Topics for Quiz 5 :

1 problem on Least Squares :

$$A\vec{x} = \vec{b} \longrightarrow A^T A \vec{x} = A^T \vec{b} .$$

1 problem on Diagonalization :

$$\vec{x}_{n+1} = A^n \vec{x}_n \quad \text{or} \quad \vec{x}\,'(t) = A\,\vec{x}(t)$$

- Find eigenvalues & eigenvectors of A.
- Express $\vec{x}_0$ or $\vec{x}(0)$ in terms of eigenvectors of A.
- Find explicit formula for $\vec{x}_n$ or $\vec{x}(t)$.