HW3 will be due Friday before class.
Will be posted before Tuesdy's class.

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Some technical definitions:

- Given vectors $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_d \in \mathbb{R}^n$, a linear combination in any vector of the form
  "$t_1 \vec{u}_1 + t_2 \vec{u}_2 + \cdots + t_d \vec{u}_d$"
  for some scalars $t_1, t_2, \ldots, t_d \in \mathbb{R}$.
- We say vectors $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_d \in \mathbb{R}^n$ are linearly dependent if there exists a linear relation

$$t_1 \vec{u}_1 + t_2 \vec{u}_2 + \cdots + t_d \vec{u}_d = \vec{0}$$

where $t_1, t_2, \ldots, t_d$ are not all zero.
Otherwise we say that $\vec{u}_1, \ldots, \vec{u}_d$ are linearly independent.

[Example: If $\vec{u}_1 = t\vec{u}_2$ for some $t$
then $\vec{u}_1 - t\vec{u}_2 = \vec{0}$ is a linear

relation, hence $\vec{u}_1$ & $\vec{u}_2$ are linearly dependent.]

Intuition: If $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_d$ are independent then they are pointing in "$d$ different directions."

- If $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_d \in \mathbb{R}^n$ are independent, then for any point $\vec{p} \in \mathbb{R}^n$ the set of points

$$\{\vec{p} + t_1\vec{u}_1 + \cdots + t_d\vec{u}_d : t_1, \ldots, t_d \in \mathbb{R}\}$$

is called a "$d$-dimensional plane" (or "$d$-plane") living in $\mathbb{R}^n$.

Common Names:

"0-plane" = "point"

"1-plane" = "line"

"2-plane" = "plane"

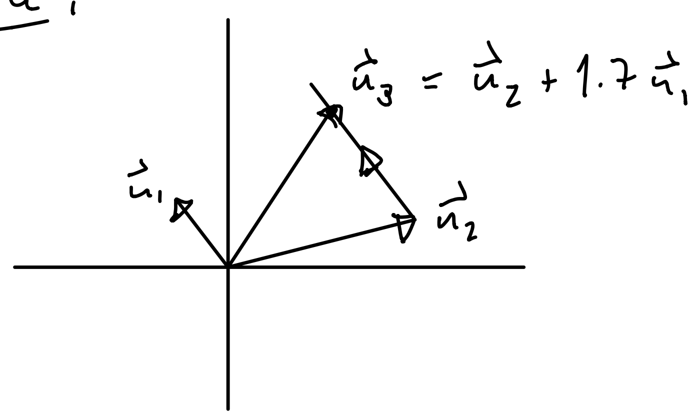
$\vdots$

"$(n-1)$-plane" = "hyperplane"

"$n$-plane" = "the whole space $\mathbb{R}^n$"

Remark: If $d > n$ then it is impossible to have an independent set of $d$ vectors in $\mathbb{R}^n$. Put another way, any set of $d$ vectors must have a linear relation among themselves.

Example:




3 vectors living in $\mathbb{R}^2$ cannot be independent! In this picture we can express any of $\vec{u}_1, \vec{u}_2, \vec{u}_3$ as a linear combination of the other two. E.g. $\vec{u}_3 = 1.7\,\vec{u}_1 + 1\,\vec{u}_2$.

This is how we know that the vector space $\mathbb{R}^2$ is "2-dimensional."

- If a d-plane in $\mathbb{R}^n$ contains the origin $\vec{0}$ then we will call it a "d-dimensional subspace of $\mathbb{R}^n$."

$$\{t_1\vec{u}_1 + \cdots + t_d\vec{u}_d : t_1, \dots, t_d \in \mathbb{R}\}$$

there is no point here.

Last week I mentioned that the solutions of m linear equations in n unknowns form a d-plane in $\mathbb{R}^n$ for some

$$0 \leq d \leq n,$$

or there is no solution. Moreover, if the equations are chosen "randomly" then we will have

$$d = n - m$$

dimension = # unknowns − # equations.

This week will discuss an <u>explicit algorithm</u> for computing these solutions

"Gaussian Elimination"

Idea: There are 3 kinds of "elementary row operations" (EROs) that we can perform on a linear system of equations.

Type I: Swap two rows/equations.

Type II: Multiply a given row/eqn. by any nonzero constant.

Type III: Replace row $R_i$ by $R_i + \ell R_j$ for some scalar $\ell$ & indices $i \neq j$.

Theorem: EROs preserve the solutions of a linear system:

$$\{\text{system 1}\} \xrightarrow{\text{EROs}} \{\text{system 2}\}$$

These two systems have exactly the

same solutions.

Proof: Easy but tedious. So we won't do it. The idea is that every ERO is invertible, i.e., can be "undone." ///

Gaussian Elimination can be described as follows:

$$\left\{ \begin{matrix} \text{original} \\ \text{system} \end{matrix} \right\} \xrightarrow{\text{sequence of EROs}} \left\{ \text{RREF} \right\}$$

The RREF ("reduced row echelon form") is the simplest possible version of the system, from which we can easily read off the solutions.

First an Example:

$$\begin{cases} x_1 + 3x_2 + 3x_3 + x_4 = 5, \\ 0 + 0 + 2x_3 + 2x_4 = 3, \\ x_1 + 3x_2 + x_3 - x_4 = 2. \end{cases}$$

Solve for $x_1, x_2, x_3, x_4$.

To save space, we will only keep track of the constants:

$$\left(\begin{array}{cccc|c} 1 & 3 & 3 & 1 & 5 \\ 0 & 0 & 2 & 2 & 3 \\ 1 & 3 & 1 & -1 & 2 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

Top left entry is our 1st pivot.

Apply EROs of Type III to eliminate all entries below the pivot

$$\left(\begin{array}{cccc|c} 1 & 3 & 3 & 1 & 5 \\ 0 & 0 & 2 & 2 & 3 \\ 0 & 0 & -2 & -2 & -3 \end{array}\right) \begin{array}{l} R_1' = R_1 \\ R_2' = R_2 \\ R_3' = R_3 - 1R_1 \end{array}$$

oops. These turned out to be zero

Find the next pivot in row 2.

Use EROs of Type III to eliminate below the pivot:

$$\left(\begin{array}{cccc|c} 1 & 3 & 3 & 1 & 5 \\ 0 & 0 & 2 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} R_1'' = R_1' \\ R_2'' = R_2' \\ R_3'' = R_3' - (-1)R_2' \end{array}$$

Look we got a row of zeroes!
This corresponds to true but useless equation $0x_1 + 0x_2 + 0x_3 + 0x_4 = 0$, so we can just ignore it (or throw it away if you like).

Now there are no more pivots to find.

Next step: Convert all pivots to 1 using EROs of type $\underline{\text{II}}$:

$$\left(\begin{array}{cccc|c} 1 & 3 & 3 & 1 & 5 \\ 0 & 0 & 1 & 1 & 3/2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \quad \begin{array}{l} R_1''' = R_1'' \\ R_2''' = \frac{1}{2} R_2'' \end{array}$$

Final Step: Apply EROs of Type $\underline{\text{III}}$ to eliminate <u>above</u> the pivots:

$$\left(\begin{array}{cccc|c} 1 & 3 & 0 & -2 & 1/2 \\ 0 & 0 & 1 & 1 & 3/2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \quad \begin{array}{l} R_1'''' = R_1''' - 3R_2''' \\ R_2'''' = R_2''' \end{array}$$

Now we are done. This is called the RREF.

Translate this back into equations:

$$\begin{cases} x_1 + 3x_2 + 0 - 2x_4 = 1/2 \\ 0 + 0 + x_3 + x_4 = 3/2 \end{cases}$$

Pivot variables: $x_1, x_3$

Free variables: $x_2, x_4$.

The solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1/2 - 3x_2 + 2x_4 \\ x_2 \\ 3/2 - x_4 \\ x_4 \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} 1/2 \\ 0 \\ 3/2 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 2 \\ 0 \\ -1 \\ 1 \end{pmatrix}}_{\text{2-plane living in } \mathbb{R}^4.}$$

Next time I'll show you how to implement Gaussian Elimination on a computer.