HW4 due Friday <u>before</u> class.

Problem 1:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\vec{x} = \vec{y} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \vec{z} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Review:

$$\underset{m \times n}{\text{Matrices}} = \underset{\mathbb{R}^n \to \mathbb{R}^m}{\text{Linear Functions}}$$

What does this <u>mean</u>?

Say $A = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}$.

This defines a function $f: \mathbb{R}^3 \to \mathbb{R}^2$ by $f(x,y,z) = A \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$= \begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + 2z \\ x + y + z \end{pmatrix}.$$

We could say:

$$f(x,y,z) = (x+2z,\ x+y+z)$$

3 inputs & 2 outputs.

Conversely, define $g: \mathbb{R}^2 \to \mathbb{R}^3$ by

$$g(x,y) = (x+y,\ x+2y,\ -x+3y)$$

Is this linear? Does it come from a matrix?

Yes: $g(x,y) = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

Two ways to think about this:

① Algebra: Check that $g(\vec{x})$ satisfies $g(s\vec{x} + t\vec{y}) = s\,g(\vec{x}) + t\,g(\vec{y})$.

Then define the matrix

$$[g] = \left(g\begin{pmatrix} 1 \\ 0 \end{pmatrix} \middle| \, g\begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ -1 & 3 \end{pmatrix} \checkmark$$

(2) Calculus:

$$g(x,y) = (u(x,y), v(x,y), w(x,y))$$
$$= (x+y,\ x+2y,\ -x+3y)$$

The total derivative Dg is defined as the matrix of partial derivatives:

$$Dg = \begin{pmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \\ \partial w/\partial x & \partial w/\partial y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ -1 & 3 \end{pmatrix}$$

What about $h(x,y,z) = (\underbrace{x^2+z}_{u},\ \underbrace{xy}_{v})$.

Is $h: \mathbb{R}^3 \to \mathbb{R}^2$ a linear function?

NO! But it still has a total derivative:

$$Dh = \begin{pmatrix} \partial u/\partial x & \partial u/\partial y & \partial u/\partial z \\ \partial v/\partial x & \partial v/\partial y & \partial v/\partial z \end{pmatrix}$$

$$= \begin{pmatrix} 2x & 0 & 1 \\ y & x & 0 \end{pmatrix}.$$

We know that h is not linear because the matrix of partials is NOT CONSTANT.

However, the function still has a "linear approximation" near any point.

Example: For $(x,y,z) \approx (1,2,3)$ we have

$$h(x,y,z) \approx h(1,2,3) + Dh(1,2,3)\begin{pmatrix} x-1 \\ y-2 \\ z-3 \end{pmatrix}$$

$$\approx \begin{pmatrix} 4 \\ 6 \end{pmatrix} + \begin{pmatrix} 2 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}\begin{pmatrix} x-1 \\ y-2 \\ z-3 \end{pmatrix}$$

Composition of Linear Functions:

$f(x,y) = (x+y,\ x)$

$g(x,y,z) = (x+z,\ 2y-x)$.

$$f(x,y) = [f]\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$g(x,y,z) = [g]\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 2 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Composite:

g has 2 outputs $\qquad g: \mathbb{R}^3 \to \mathbb{R}^2$

f has 2 inputs $\qquad f: \mathbb{R}^2 \to \mathbb{R}^2$

So we can consider the composite

$$f \circ g : \mathbb{R}^3 \to \mathbb{R}^2.$$

$$f(g(x,y,z)) = f(x+z,\ 2y-x)$$
$$= \left((x+\cancel{z}) + (2y-\cancel{z}),\ (x+z) \right)$$
$$= (x+2y,\ x+z).$$

The composite is linear with matrix

$$[f \circ g] = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

OR: Matrix Multiplication

$$[f][g] = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ -1 & 2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 2 & 1 \\ 1 & 0 & 1 \end{pmatrix} \quad \text{SAME} \ \checkmark$$

Theorem: If $f : \mathbb{R}^\ell \to \mathbb{R}^m$ and $g : \mathbb{R}^n \to \mathbb{R}^\ell$ are linear, then the composite $f \circ g : \mathbb{R}^n \to \mathbb{R}^m$ is linear and the matrices satisfy:

$[f \circ g] = [f][g]$

composition = matrix multiplication.

[Remark: In Calculus, this is known as the higher dimensional "chain rule":

$D(f \circ g) = Df \cdot Dg$.]

How to compute the matrix product?

A $m \times l$ $\rightsquigarrow$ AB $m \times n$.
B $l \times n$

Many ways to think about it:

- (ij entry AB) = (ith row A)(jth col B)
  dot product.

Let $A = (a_{ij})$, $B = (b_{ij})$, $AB = (c_{ij})$.

Then

$$c_{ij} = \sum_{k=1}^{l} a_{ik} b_{kj}.$$

- (jth column AB) = A(jth col B)
- (ith row AB) = (ith row A) B.

Finally,

- $AB = \sum_{k=1}^{\ell} (k\text{th col } A)(k\text{th row } B)$.

Example:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 2 & 3 \end{pmatrix}.$$

2nd row of AB

= (2nd row A) B

$$= \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 8 & 10 \end{pmatrix}$$

3rd column of AB

= A (3rd column of B)

$$= \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 10 \end{pmatrix}$$

Finally,

$$AB = \sum_{k=1}^{3} (k\text{th col } A)(k\text{th row } B)$$

$$= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} \begin{pmatrix} -1 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \end{pmatrix} \begin{pmatrix} 0 & 2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 1 & 0 \\ -2 & 2 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 2 & 3 \\ 0 & 6 & 9 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 3 & 4 \\ -1 & 8 & 10 \end{pmatrix} \quad \checkmark$$

Almost TOO MANY ways to think about matrix multiplication!!

[Remark: If $\vec{x}, \vec{y} \in \mathbb{R}^n$

$\vec{x}^T \vec{y} = \vec{y}^T \vec{x}$ is 1×1 (number)

called the dot product. If

$\vec{x} \in \mathbb{R}^m$, $\vec{y} \in \mathbb{R}^n$ then

$\vec{x}\,\vec{y}^{T}$ is $m \times n$ matrix

$\underbrace{m \times 1 \quad 1 \times n}$

$\vec{y}\,\vec{x}^{T}$ is $n \times m$ matrix.

$\underbrace{n \times 1 \quad 1 \times m}$

Jargon: Matrices of the form

(column)(row)

are called "rank 1 matrices" because every matrix of rank 1 has this form!

Summary: Matrix Arithmetic.

- Matrices of the same shape can be added & multiplied by scalars:

$$\begin{array}{lll} A, B \quad m \times n & \rightsquigarrow & A+B \quad m \times n \\ t \in \mathbb{R} & & tA \quad m \times n. \end{array}$$

- These operations make the set of $m \times n$ matrices into an "abstract vector space":

$$A + B = B + A$$
$$A + (B + C) = (A + B) + C$$
$$\vdots$$
etc.

- Furthermore, if # cols A = # rows B then we have a product matrix AB.
- Matrix products satisfy the following additional rules:

$$A(BC) = (AB)C$$
$$A(B+C) = AB + AC$$
$$(A+B)C = AC + BC$$
$$t(AB) = (tA)B = A(tB)$$

$$\boxed{\begin{array}{c}\text{WARNING:}\\ AB \neq BA\end{array}}$$

- Finally, the transpose $A \mapsto A^T$ satisfies the following rules:

$$(A+B)^T = A^T + B^T$$
$$(tA)^T = t(A^T)$$
$$(AB)^T = B^T A^T$$

Note that matrix arithmetic includes vector arithmetic as a special case because:

vectors are matrices
dot product is matrix multiplication

New Topic:

Invertibility of Matrices.

Definition: We say that matrix A is invertible if there exists some matrix B such that

$$\begin{cases} AB = I \\ BA = I \end{cases}$$

In this case the matrix B is unique.
Indeed, suppose we had two inverses:

$$\begin{cases} AB = I \\ BA = I \end{cases} \quad \& \quad \begin{cases} AC = I \\ CA = I \end{cases}.$$

Then we must have

$$B = BI = B(AC)$$
$$= (BA)C = IC = C.$$

The unique inverse matrix (if it exists) is called "A^{-1}".

TWO QUESTIONS:

① How can we tell when a matrix is invertible?

② How can we <u>compute</u> the inverse?

NEXT TIME.

For now, some examples.

$$F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \text{reflect across } x=y.$$

Invertible?

We want to find some matrix F^{-1} such that

$$FF^{-1} = I \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} F^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$F^{-1}F = I \qquad F^{-1}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Conceptually: F^{-1} is the linear function that "undoes F" or "does the opposite of F."

Claim: F^{-1} exists and $F^{-1} = F$.

Indeed, to undo a reflection, we can just perform the same reflection again!

Check: $$FF = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I \quad \checkmark$$

Next: $R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ = rotate c.c.w. 90°

R^{-1} = the opposite function

= rotate c.w. 90°

= rotate c.c.w. 90°, three times

$= R^3$

$= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Teaser: 2×2 matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Inverse exists

$\iff ad - bc \neq 0$

$\iff \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0$.