HW5 Solutions

Problem 1: To fit a parabola $y = a + bx + cx^2$ to four data points:

$(x, y) = (1,1), (2,4), (3,3), (4,2)$.

$$\begin{cases} a + b + c = 1 \\ a + 2b + 4c = 4 \\ a + 3b + 9c = 3 \\ a + 4b + 16c = 2 \end{cases} \rightsquigarrow \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 3 \\ 2 \end{pmatrix}$$

$$X \vec{a} = \vec{y}.$$

This has no solution so instead we solve the normal equation

$$X^T X \vec{a} = X^T \vec{y}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 16 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 16 \end{pmatrix} \begin{pmatrix} 1 \\ 4 \\ 3 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 10 & 30 \\ 10 & 30 & 100 \\ 30 & 100 & 354 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 10 \\ 26 \\ 76 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 4 & 10 & 30 & 10 \\ 10 & 30 & 100 & 26 \\ 30 & 100 & 354 & 76 \end{array}\right)$$

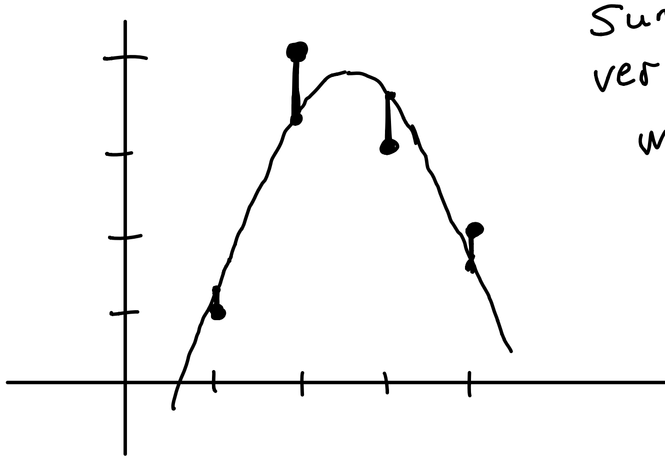
$$\xrightarrow{\text{RREF}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 26/5 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

I used a computer for this.

Hence $(a, b, c) = \left(-3, \frac{26}{5}, -1\right)$ and the best fit parabola is

$$y = -3 + \frac{26}{5}x - x^2.$$

Here is a picture:



Sum of squares of vertical errors is minimized.

Problem 2:

a) $P = \vec{a}(\vec{a}^T\vec{a})^{-1}\vec{a}^T$

$$= \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \left[\begin{pmatrix} 2 & -3 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \right]^{-1} \begin{pmatrix} 2 & -3 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} (14)^{-1} \begin{pmatrix} 2 & -3 & 1 \end{pmatrix}$$

$$= \frac{1}{14} \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \begin{pmatrix} 2 & -3 & 1 \end{pmatrix}$$

$$= \frac{1}{14} \begin{pmatrix} 4 & -6 & 2 \\ -6 & 9 & -3 \\ 2 & -3 & 1 \end{pmatrix}$$

b) $Q = A(A^TA)^{-1}A^T$

$$= \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 4 \end{pmatrix} \left[\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 4 \end{pmatrix} \right]^{-1} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \end{pmatrix}$$

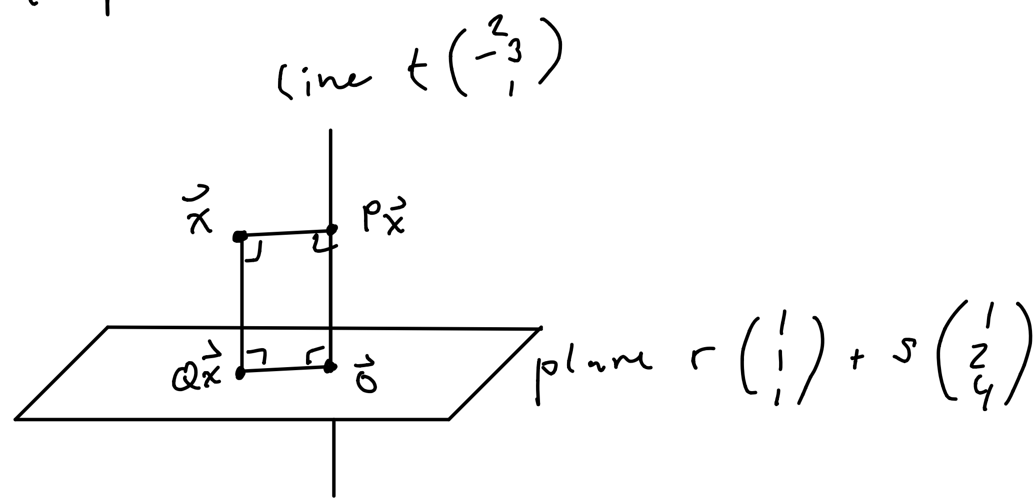
$$= \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 3 & 7 \\ 7 & 21 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 4 \end{pmatrix} \frac{1}{14} \begin{pmatrix} 21 & -7 \\ -7 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \end{pmatrix}$$

$$= \frac{1}{14}\begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} 14 & 7 & -7 \\ -4 & -1 & 5 \end{pmatrix}$$

$$= \frac{1}{14}\begin{pmatrix} 10 & 6 & -2 \\ 6 & 5 & 3 \\ -2 & 3 & 13 \end{pmatrix}$$

c) We have $P + Q = I$ because P & Q project onto complementary subspaces:



$$\vec{x} = P\vec{x} + Q\vec{x} = (P+Q)\vec{x}.$$

for all $\vec{x}$ implies that $P + Q = I$.

Problem 3: Let $c = \cos\theta$, $s = \sin\theta$.

$$R_\theta = \begin{pmatrix} c & -s \\ s & c \end{pmatrix}, \quad F_\theta = \begin{pmatrix} c & s \\ s & -c \end{pmatrix}, \quad P_\theta = \begin{pmatrix} c^2 & cs \\ cs & s^2 \end{pmatrix}.$$

a) R_θ rotates c.c.w. by θ.

F_θ reflects across line making angle $\theta/2$ from x-axis

P_θ projects onto line making angle θ from x-axis.

b) $\det(R_\theta) = c^2 + s^2 = 1.$

$\det(F_\theta) = -c^2 - s^2 = -1.$

$\det(P_\theta) = c^2 s^2 - (cs)^2 = 0.$

c) $R_\theta^{-1} = \frac{1}{1}\begin{pmatrix} c & s \\ -s & c \end{pmatrix} \quad (= R_{-\theta})$

$F_\theta^{-1} = \frac{1}{-1}\begin{pmatrix} -c & -s \\ -s & c \end{pmatrix} \quad (= F_\theta)$

P_θ^{-1} does not exist.

[Recall: A^{-1} exists $\Longleftrightarrow \det(A) \neq 0$.]

Problem 4:

$$A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}.$$

a) Eigenvalues:

$$\det(A - \lambda I) = 0$$

$$\det\begin{pmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{pmatrix} = 0$$

$$(1-\lambda)(3-\lambda) - 2\cdot 4 = 0$$

$$\lambda^2 - 4\lambda + 3 - 8 = 0$$

$$\lambda^2 - 4\lambda + 5 = 0$$

$$\lambda = -1 \text{ or } 5.$$

b) (-1)-eigenspace:

$$(A + 1I)\vec{u} = \vec{0}.$$

$$\begin{pmatrix} 2 & 2 \\ 4 & 4 \end{pmatrix}\vec{u} = \vec{0}$$

$$\vec{u} = t\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

5-eigenspace:

$$(A - 5I)\vec{v} = \vec{0}$$

$$\begin{pmatrix} -4 & 2 \\ 4 & -2 \end{pmatrix} \vec{v} = \vec{0}$$

$$\vec{v} = t \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

c) A^n has eigenvalues $(-1)^n, 5^n$ with the same eigenvectors:

$$A^n \begin{pmatrix} 1 \\ -1 \end{pmatrix} = (-1)^n \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$A^n \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 5^n \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

e^{At} has eigenvalues e^{-t}, e^{5t} with the same eigenvectors:

$$e^{At} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$e^{At} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = e^{5t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Problem 5:

a) $\begin{pmatrix}5\\4\end{pmatrix} = 2\begin{pmatrix}1\\-1\end{pmatrix} + 3\begin{pmatrix}1\\2\end{pmatrix}$

b) The general solution of

$$\begin{pmatrix}x_0\\y_0\end{pmatrix} = \begin{pmatrix}5\\4\end{pmatrix} \quad \& \quad \begin{pmatrix}x_{n+1}\\y_{n+1}\end{pmatrix} = A\begin{pmatrix}x_n\\y_n\end{pmatrix}$$

is
$$\begin{aligned}
\begin{pmatrix}x_n\\y_n\end{pmatrix} &= A^n\begin{pmatrix}x_0\\y_0\end{pmatrix}\\
&= A^n\begin{pmatrix}5\\4\end{pmatrix}\\
&= A^n\left[2\begin{pmatrix}1\\-1\end{pmatrix} + 3\begin{pmatrix}1\\2\end{pmatrix}\right]\\
&= 2A^n\begin{pmatrix}1\\-1\end{pmatrix} + 3A^n\begin{pmatrix}1\\2\end{pmatrix}\\
&= 2(-1)^n\begin{pmatrix}1\\-1\end{pmatrix} + 3\cdot 5^n\begin{pmatrix}1\\2\end{pmatrix}\\
&= \begin{pmatrix}3\cdot 5^n + 2(-1)^n\\ 6\cdot 5^n - 2(-1)^n\end{pmatrix}.
\end{aligned}$$

c) The general solution of

$$\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \end{pmatrix} \quad \& \quad \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = A \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

is
$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{At} \begin{pmatrix} x(0) \\ y(0) \end{pmatrix}$$

$$= e^{At} \begin{pmatrix} 5 \\ 4 \end{pmatrix}$$

$$= e^{At} \left[2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + 3 \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right]$$

$$= 2e^{At} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + 3e^{At} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$= 2e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + 3e^{5t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 3e^{5t} + 2e^{-t} \\ 6e^{5t} - 2e^{-t} \end{pmatrix}.$$

We will draw a picture of this solution in class.

Preview: