Math 210 **Summer 2020**
Homework 5 **Drew Armstrong**

Problem 1. Least Squares Approximation. We want to find the parabola $y = a+bx+cx^2$ that is a best fit for the data points $(x,y) = (1,1)$, $(2,4)$, $(3,3)$, $(4,2)$.

(a) Let $\mathbf{a} = (a,b,c)$ be the unknown parameters. Set up the equation $X\mathbf{a} = \mathbf{y}$ that would be true if all four points were on the parabola. This equation has no solution.
(b) Solve the normal equation $X^TX\mathbf{a} = X^T\mathbf{y}$ to find the (OLS) best values of a,b,c.
(c) Draw the parabola and the data points.

Problem 2. Complementary Projections. Consider the following matrices:

$$\mathbf{a} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 4 \end{pmatrix}.$$

(a) Compute the matrix $P = \mathbf{a}(\mathbf{a}^T\mathbf{a})^{-1}\mathbf{a}^T$.
(b) Compute the matrix $Q = A(A^TA)^{-1}A^T$.
(c) Check that $P + Q = I$. Why does this happen?

Problem 3. Special Matrices. Consider the following matrices:

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad F_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}, \quad P_\theta = \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix}.$$

(a) Describe what each matrix does geometrically.
(b) Compute the determinant of each matrix.
(c) For each matrix that is invertible, compute the inverse.

Problem 4. Eigenvalues and Eigenvectors. Consider the following matrix:

$$A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}.$$

(a) Solve the characteristic equation $\det(A - \lambda I) = 0$ to find the eigenvalues.
(b) For each eigenvalue find all of the corresponding eigenvectors.
(c) Find the eigenvalues and eigenvectors of the matrices A^n and e^{At}. [Hint: You don't need to do any more work.]

Problem 5. Diagonalization. Let A be the same matrix from Problem 4.

(a) Express the vector $(5,4)$ as a linear combination of eigenvectors of A.
(b) Suppose the numbers x_n and y_n are defined as follows:

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = A\begin{pmatrix} x_n \\ y_n \end{pmatrix}.$$

Use part (a) and Problem 4 to find explicit formulas for x_n and y_n. [Recall that the general solution looks like $\mathbf{x}_n = a\lambda^n\mathbf{u} + b\mu^n\mathbf{v}$.]
(c) Suppose the functions $x(t)$ and $y(t)$ are defined as follows:

$$\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = A\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

Use part (a) and Problem 4 to find explicit formulas for $x(t)$ and $y(t)$. [Recall that the general solution looks like $\mathbf{x}(t) = ae^{\lambda t}\mathbf{u} + be^{\mu t}\mathbf{v}$.]