HW4 Solutions

Problem 1: $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}$

$\vec{x} = \vec{y} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\vec{z} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

(a) $AB z = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$= \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \end{pmatrix}$

(b) $A(\vec{x} + \vec{y}) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \left[\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right]$

$= \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \end{pmatrix}$

(c) $\vec{z}^T A \vec{x} = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 3 \end{pmatrix} = 6$

(d) $BA + \vec{x}\vec{y}^T$

$$= \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$$

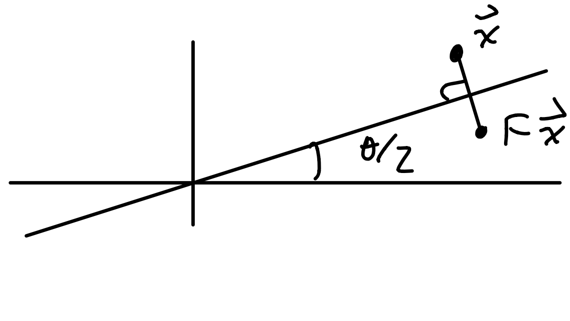
$$= \begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{pmatrix}.$$

Problem 2: Many correct answers.

(a) Any matrix $F = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$ satisfies $F \neq I$ & $F^2 = I$.

[Geometrically, this matrix reflects across the line with angle $\theta/2$:



we will discuss this in class.]

Three Easy Examples :

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ (\theta = 0), \quad \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \ (\theta = 180^\circ), \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ (\theta = 90^\circ)$$

(b) Let $R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ = rotation c.c.w. by 90°.

Then $R \neq I$

$$R^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \neq I$$

$$R^3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \neq I$$

$$R^4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I.$$

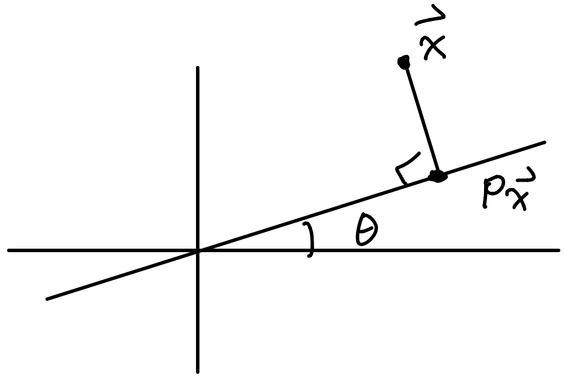
We could also take

$$R = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \text{rotation c.w. by } 90^\circ.$$

(c) Any matrix $P = \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix}$

satisfies $P \neq I$, $P^2 = P$.

[Geometrically, this matrix <u>projects</u> onto the line of angle θ;



We will discuss this in class.]

Two Easy Examples:

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{or} \quad P = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

= project onto x-axis

= project onto y-axis

Problem 3: $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 1 & 2 & 4 \end{pmatrix}$

(a) Compute A^{-1}:

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 3 & 4 & 0 & 1 & 0 \\ 1 & 2 & 4 & 0 & 0 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -2 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 4 & 0 & -3 \\ 0 & 1 & 0 & 4 & -1 & -2 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -4 & 2 & 1 \\ 0 & 1 & 0 & 4 & -1 & -2 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right)$$

Conclusion: $A^{-1} = \begin{pmatrix} -4 & 2 & 1 \\ 4 & -1 & -2 \\ -1 & 0 & 1 \end{pmatrix}$.

(b) $A\vec{x} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

$$\Rightarrow \vec{x} = A^{-1}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\vec{x} = \begin{pmatrix} -4 & 2 & 1 \\ 4 & -1 & -2 \\ -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \\ -1 \end{pmatrix}$$

$$A\vec{y} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \Rightarrow \vec{y} = \begin{pmatrix} -4 & 2 & 1 \\ 4 & -1 & -2 \\ -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}.$$

Problem 4 :

(a) If A^{-1} exists then

$$A\vec{x} = A\vec{y}$$
$$\Rightarrow A^{-1}A\vec{x} = A^{-1}A\vec{y}$$
$$I\vec{x} = I\vec{y}$$
$$\vec{x} = \vec{y} \qquad \checkmark$$

(b) If $A\vec{x} = \vec{0}$ and $\vec{x} \neq \vec{0}$ then

A^{-1} does not exist.

Proof : If it did, then we would get a contradiction:

$$A\vec{x} = \vec{0}$$
$$A^{-1}A\vec{x} = A^{-1}\vec{0}$$
$$I\vec{x} = \vec{0}$$
$$\vec{x} = \vec{0} \quad \text{OOPS!} \qquad ///$$

(c) If A^{-1} exists then $(A^T)^{-1}$ exists.

Proof : I claim that $(A^T)^{-1} = (A^{-1})^T$.

To show this we will use the identity

$$(*) \quad (CD)^T = D^T C^T.$$

Putting $C = A$ & $D = A^{-1}$ gives

$$(A^{-1})^T A^T \overset{(*)}{=} (AA^{-1})^T = I^T = I \quad \checkmark$$

and putting $C = A^{-1}$ & $D = A$ gives

$$A^T (A^{-1})^T \overset{(*)}{=} (A^{-1}A)^T = I^T = I \quad \checkmark$$

(d) If A^{-1}, B^{-1}, AB exist then I claim that $(AB)^{-1}$ exists.

Proof : I claim that $(AB)^{-1} = B^{-1}A^{-1}$.

Check :

$$\begin{aligned}(AB)(B^{-1}A^{-1}) &= A(BB^{-1})A^{-1} \\ &= AIA^{-1} \\ &= AA^{-1} \\ &= I \quad \checkmark\end{aligned}$$

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B$$
$$= B^{-1} I B$$
$$= B^{-1}B$$
$$= I \quad \checkmark$$

Problem 5: $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 3 \end{pmatrix}$

(a) $AA^T = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}$

$$= \begin{pmatrix} 2 & 3 & 5 \\ 3 & 5 & 8 \\ 5 & 8 & 13 \end{pmatrix}$$

Note that $\begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} + \begin{pmatrix} 3 \\ 5 \\ 8 \end{pmatrix} - \begin{pmatrix} 5 \\ 8 \\ 13 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

In other words, we have

$$\begin{pmatrix} 2 & 3 & 5 \\ 3 & 5 & 8 \\ 5 & 8 & 13 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Since $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ it follows from 4b that $\begin{pmatrix} 2 & 3 & 5 \\ 3 & 5 & 8 \\ 5 & 8 & 13 \end{pmatrix}^{-1}$ does not exist.

(b) $A^T A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 3 \end{pmatrix}$

$$= \begin{pmatrix} 6 & 9 \\ 9 & 14 \end{pmatrix}.$$

Invert:

$$\left(\begin{array}{cc|cc} 6 & 9 & 1 & 0 \\ 9 & 14 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 3/2 & 1/6 & 0 \\ 9 & 14 & 0 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{cc|cc} 1 & 3/2 & 1/6 & 0 \\ 0 & 1/2 & -3/2 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 3/2 & 1/6 & 0 \\ 0 & 1 & -3 & 2 \end{array}\right)$$

$$\to \left(\begin{array}{cc|cc} 1 & 0 & 14/3 & -3 \\ 0 & 1 & -3 & 2 \end{array}\right)$$

$$\Rightarrow (A^T A)^{-1} = \begin{pmatrix} 14/3 & -3 \\ -3 & 2 \end{pmatrix}.$$

(c) $P = A(A^TA)^{-1}A^T$

$$= \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 14/3 & -3 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}$$

$$= \frac{1}{3} \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 14 & -9 \\ -9 & 6 \end{pmatrix} \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}$$

$$= \frac{1}{3} \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 5 & -4 & 1 \\ -3 & 3 & 0 \end{pmatrix}$$

$$= \frac{1}{3} \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

We will discuss the geometric interpretation next week.

Preview: