Math 210 **Summer 2020**
Homework 4 **Drew Armstrong**

Problem 1. Matching Shapes. Let A be a 2×3 matrix and let B be a 3×2 matrix. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$ and $\mathbf{z} \in \mathbb{R}^2$. Suppose that all of the entries of these matrices and vectors are equal to 1. Compute the following matrices or say why they don't exist:

(a) $AB\mathbf{z}$,
(b) $A(\mathbf{x} + \mathbf{y})$,
(c) $\mathbf{z}^T A\mathbf{x}$,
(d) $BA + \mathbf{x}\mathbf{y}^T$.

Problem 2. Special Matrices. Find specific matrices with the following properties:

(a) Find some 2×2 matrix F with $F \neq I$ and $F^2 = I$.
(b) Find some 2×2 matrix R with $R, R^2, R^3 \neq I$ and $R^4 = I$.
(c) Find some 2×2 matrix P with $P \neq 0, I$ and $P^2 = P$.

Problem 3. Computing the Inverse. Consider the following matrix:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 1 & 2 & 4 \end{pmatrix}.$$

(a) Compute the RREF of $(A|I)$ to obtain the inverse A^{-1}.
(b) Use your answer from (a) to solve the following linear systems:

$$A\mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad A\mathbf{y} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

Problem 4. Invertibility. Prove the following statements:

(a) If A^{-1} exists then $A\mathbf{x} = A\mathbf{y}$ implies $\mathbf{x} = \mathbf{y}$.
(b) If $A\mathbf{x} = \mathbf{0}$ for some $\mathbf{x} \neq \mathbf{0}$ then A^{-1} does not exist. [Hint: Use part (a).]
(c) If A^{-1} exists then $(A^T)^{-1}$ exists. [Hint: Show that $A^T(A^{-1})^T = I$.]
(d) If A^{-1}, B^{-1} and AB exist then $(AB)^{-1}$ exists. [Hint: Show that $(AB)(B^{-1}A^{-1}) = I$.]

Problem 5. A Projection Matrix. Consider the following 3×2 matrix:

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 3 \end{pmatrix}.$$

(a) Compute AA^T and explain why $(AA^T)^{-1}$ does not exist. [Hint: Find some nonzero vector $\mathbf{x} \neq \mathbf{0}$ such that $(AA^T)\mathbf{x} = \mathbf{0}$. Then use Problem 4(b).]
(b) Compute $A^T A$ and $(A^T A)^{-1}$.
(c) Compute $P = A(A^T A)^{-1} A^T$. [Remark: For any point $\mathbf{x} \in \mathbb{R}^3$, the point $P\mathbf{x} \in \mathbb{R}^3$ is the "orthogonal projection" of $\mathbf{x}$ onto the column space of A (which is a plane).]