HW3 Solutions

Problems 1 & 2 in the lecture notes.

Problem 3: Let A be $m \times n$ matrix with column vectors $\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n \in \mathbb{R}^m$. Then for any vectors $\vec{x}, \vec{y} \in \mathbb{R}^n$ and scalars $s, t \in \mathbb{R}$ we have

$$A(s\vec{x} + t\vec{y}) = \sum_{i=1}^{n} (i\text{th entry of } s\vec{x} + t\vec{y})\, \vec{a}_i$$

$$= \sum_{i=1}^{n} (s x_i + t y_i)\, \vec{a}_i$$

$$= \sum_{i=1}^{n} (s x_i \vec{a}_i + t y_i \vec{a}_i)$$

$$= s \sum_{i=1}^{n} x_i \vec{a}_i + t \sum_{i=1}^{n} y_i \vec{a}_i$$

$$= s(A\vec{x}) + t(A\vec{y}).$$

This means we can treat the operation "$A\vec{x}$" as some kind of "multiplication."

Problem 4: If $\vec{x} = (x_1, x_2, x_3, x_4)$ is $\perp$ to $\vec{u} = (1,1,1,1)$ and $\vec{v} = (1,2,3,4)$, then we must have

$$\begin{cases} \vec{u} \cdot \vec{x} = 0 \\ \vec{v} \cdot \vec{x} = 0 \end{cases}$$

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ x_1 + 2x_2 + 3x_3 + 4x_4 = 0 \end{cases}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 4 & 0 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 3 & 0 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 0 & -1 & -2 & 0 \\ 0 & 1 & 2 & 3 & 0 \end{array}\right)$$

$$\begin{cases} x_1 \quad - x_3 - x_4 = 0 \\ \quad x_2 + 2x_3 + 3x_4 = 0 \end{cases}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_3 + 2x_4 \\ -2x_3 - 3x_4 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix}$$

If are looking for some special vector "$\vec{u} \times \vec{v}$" with the properties

$$\vec{u} \cdot (\vec{u} \times \vec{v}) = 0$$

$$\vec{v} \cdot (\vec{u} \times \vec{v}) = 0$$

then we won't find one because there is a whole plane of such vectors.

Problem 5: Let $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_d \in \mathbb{R}^n$ be linearly independent and consider the $d \times n$ matrix

$$A = \underbrace{\left.\begin{pmatrix} \text{---}\ \vec{u}_1^T\ \text{---} \\ \text{---}\ \vec{u}_2^T\ \text{---} \\ \vdots \\ \text{---}\ \vec{u}_d^T\ \text{---} \end{pmatrix}\right\} d}_{n}$$

Call the row space

$$U = R(A) = \{ t_1 \vec{u}_1 + \cdots + t_d \vec{u}_d \} \subseteq \mathbb{R}^n.$$

Then the orthogonal complement is the nullspace:

$$
\begin{aligned}
U^\perp &= N(A) \\
&= \{\vec{x} \in \mathbb{R}^n : A\vec{x} = \vec{0}\} \\
&= \{\vec{x} \in \mathbb{R}^n : \vec{u}_i \cdot \vec{x} = 0 \text{ for all } i\}
\end{aligned}
$$

We want to show that

$$\dim U^\perp = \dim N(A) = n - d.$$

Recall that

$$
\begin{aligned}
\dim N(A) &= \#\text{ nonpivot columns in } RREF(A). \\
&= \#\text{ columns} - \#\text{ pivot columns} \\
&= n - \#\text{ pivot columns}.
\end{aligned}
$$

So our job is to show that

$$\#\text{pivots in } RREF(A) = d$$

i.e.,

$$\#\text{ pivots in } RREF(A) = \#\text{rows of } A.$$

I claim that this is true because the rows of A are assumed to be linearly independent. Indeed, we observe that every row of RREF(A) is a linear combination of the rows of A (and vice versa). If the rows of A are independent this means that we can never get a row of zeroes in RREF(A), meaning there will be a pivot in every row. ///

[In general, the number of pivots in RREF(A) tells us the "number of linearly independent rows in A."]