Math 210 **Summer 2020**
Homework 3 **Drew Armstrong**

Problem 1. Use Gaussian Elimination to put the following matrix in RREF:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \end{pmatrix}$$

Show your work!

Problem 2. Use your answer from Problem 1 to solve the following system of linear equations:

$$\begin{cases} x & + & 2y & + & 3z & = & 4 \\ 2x & + & 3y & + & 4z & = & 5 \\ 3x & + & 4y & + & 5z & = & 6 \end{cases}$$

Problem 3. Matrices are Linear Functions. Let A be an $m \times n$ matrix with column vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n \in \mathbb{R}^m$. Then for any $n \times 1$ column vector $\mathbf{x} \in \mathbb{R}^n$ we define an $m \times 1$ column vector "$A\mathbf{x}$" $\in \mathbb{R}^m$ by the following formula:

$$\begin{aligned} A\mathbf{x} &= x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n \\ &= (\text{1st entry } \mathbf{x})(\text{1st column } A) + \cdots + (n\text{th entry } \mathbf{x})(n\text{th column } A). \end{aligned}$$

Use this definition to prove that for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $s, t \in \mathbb{R}$ we have

$$A(s\mathbf{x} + t\mathbf{y}) = s(A\mathbf{x}) + t(A\mathbf{y}).$$

Problem 4. There is no Cross Product in 4D. Find all of the vectors in $\mathbb{R}^4$ that are simultaneously perpendicular to $(1, 1, 1, 1)$ and $(1, 2, 3, 4)$. Use your answer to explain why there is no such thing as the "cross product" in $\mathbb{R}^4$.

Problem 5. Orthogonal Complement. Let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_d \in \mathbb{R}^n$ be an independent set of vectors living in n-dimensional space. These vectors define a d-dimensional subspace of $\mathbb{R}^n$:

$$U = \{t_1\mathbf{u}_1 + t_2\mathbf{u}_2 + \cdots + t_d\mathbf{u}_d : t_1, t_2, \ldots, t_d \in \mathbb{R}\} \subseteq \mathbb{R}^n.$$

The *orthogonal complement of* U is defined as the set of vectors that are simultaneously perpendicular to every vector in U:

$$U^\perp = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{u}_i \bullet \mathbf{x} = 0 \text{ for all } i = 1, 2 \ldots, d\} \subseteq \mathbb{R}^n.$$

Explain why $U^\perp$ is an $(n - d)$-dimensional subspace of $\mathbb{R}^n$. [Hint: Consider the RREF of the system of equations $\mathbf{u}_i \bullet \mathbf{x} = 0$. How many pivot variables and free variables does it have?]