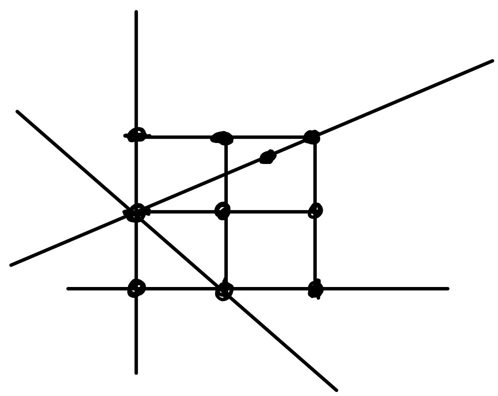
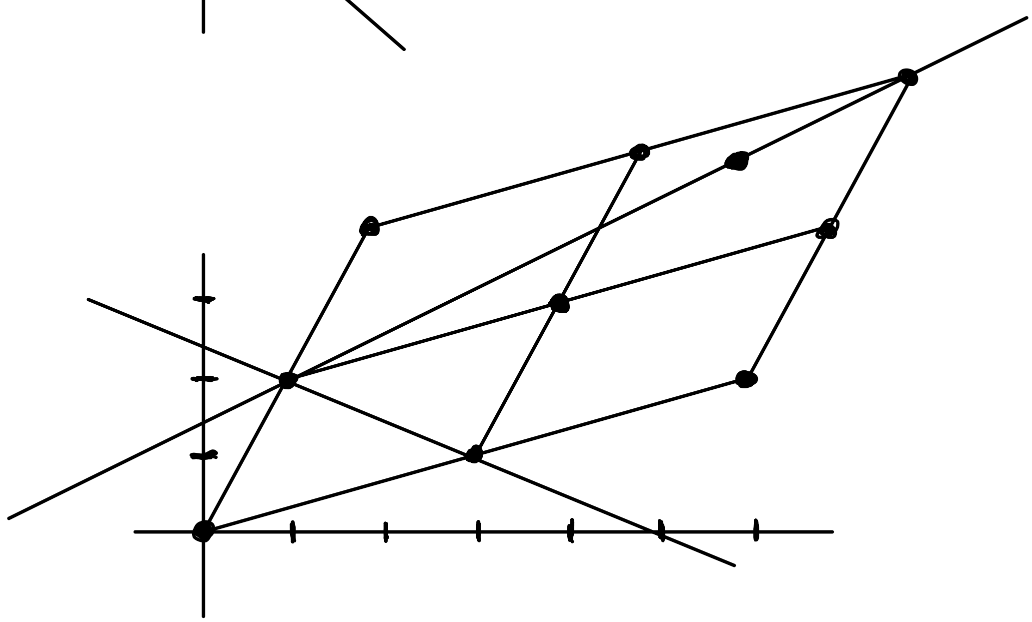
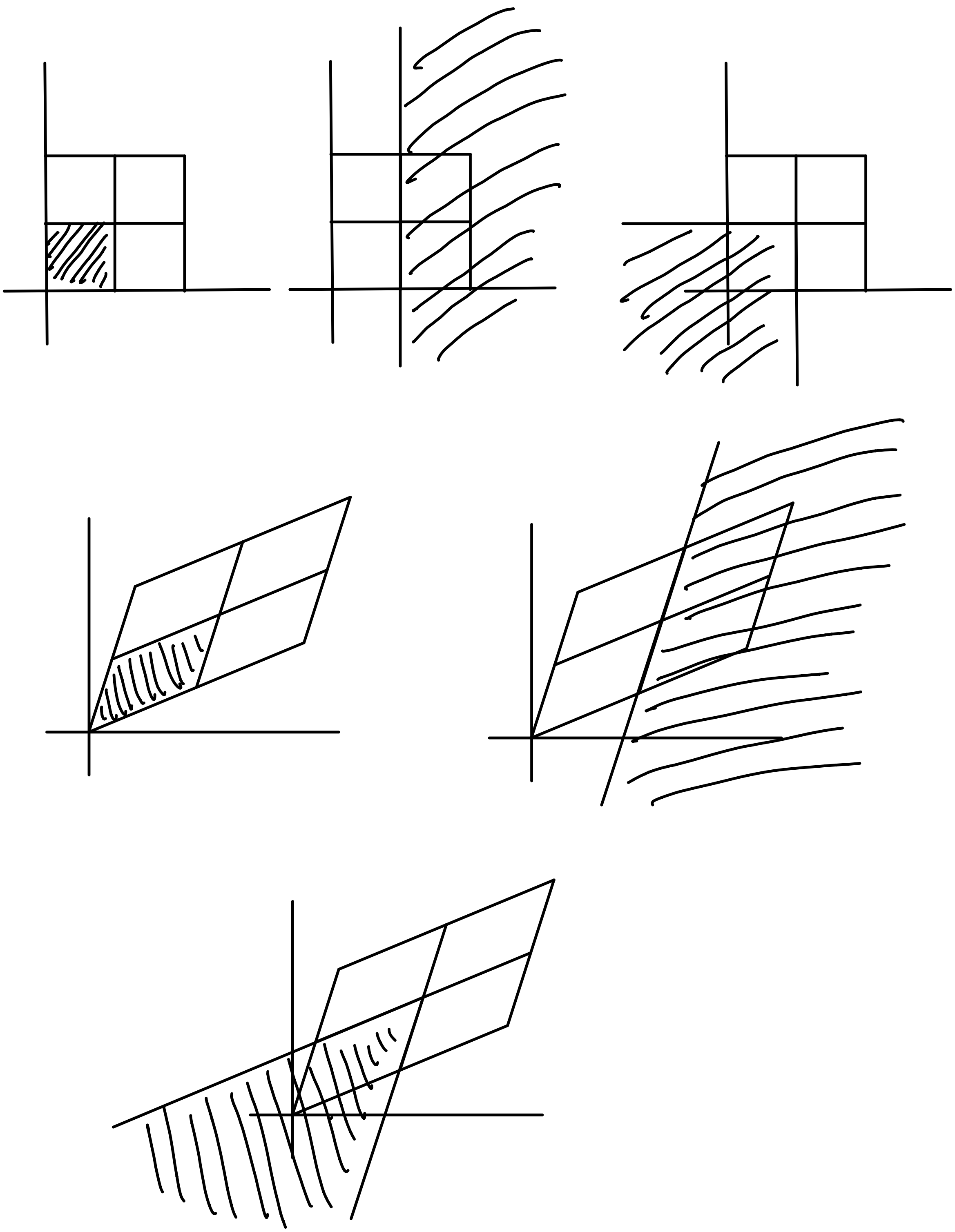
HW1 Solutions

Problems 1 & 2 :



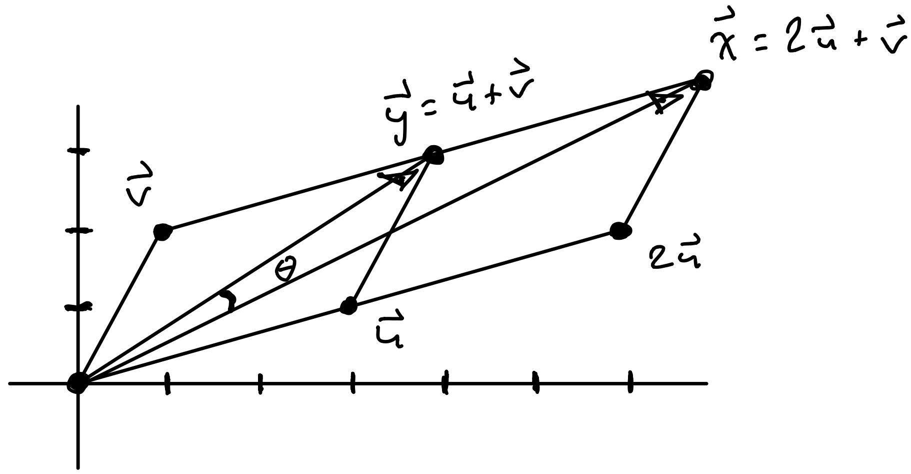


Problem 3:



Problem 4:

(a)



$\vec{x} = (7,4)$ & $\vec{y} = (4,3)$

$\vec{x} \circ \vec{x} = 7^2 + 4^2 = 65$

$\vec{y} \circ \vec{y} = 4^2 + 3^2 = 25$

$\vec{x} \circ \vec{y} = 28 + 12 = 40$

$$\cos\theta = \frac{\vec{x} \circ \vec{y}}{\sqrt{\vec{x} \circ \vec{x}}\sqrt{\vec{y} \circ \vec{y}}} = \frac{40}{\sqrt{65}\sqrt{25}}$$

$$\theta = \arccos\left(\frac{40}{\sqrt{65}\sqrt{25}}\right) \approx 7.1^\circ$$

(b) Now consider any $\vec{u}, \vec{v} \in \mathbb{R}^{100}$ with $\vec{u} \circ \vec{v} = 5$, $\vec{u} \circ \vec{u} = 10$, $\vec{v} \circ \vec{v} = 5$

and define $\vec{x} = 2\vec{u} + \vec{v}$

$\vec{y} = \vec{u} + \vec{v}$.

Let θ be the angle between $\vec{x}$ & $\vec{y}$.

Compute:

$$\vec{x} \circ \vec{x} = (2\vec{u} + \vec{v}) \circ (2\vec{u} + \vec{v})$$
$$= 4\vec{u} \circ \vec{u} + 4\vec{u} \circ \vec{v} + 1\vec{v} \circ \vec{v}$$
$$= 4(10) + 4(5) + 1(5) = 65$$

$$\vec{y} \circ \vec{y} = (\vec{u} + \vec{v}) \circ (\vec{u} + \vec{v})$$
$$= \vec{u} \circ \vec{u} + 2\vec{u} \circ \vec{v} + \vec{v} \circ \vec{v}$$
$$= (10) + 2(5) + (5) = 25$$

$$\vec{x} \cdot \vec{y} = (2\vec{u} + \vec{v}) \circ (\vec{u} + \vec{v})$$
$$= 2\vec{u} \circ \vec{u} + 3\vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{v}$$
$$= 2(10) + 3(5) + (5) = 40$$

SAME AS BEFORE!

Hence $\theta \approx 7.1°$.

Problem 5 :

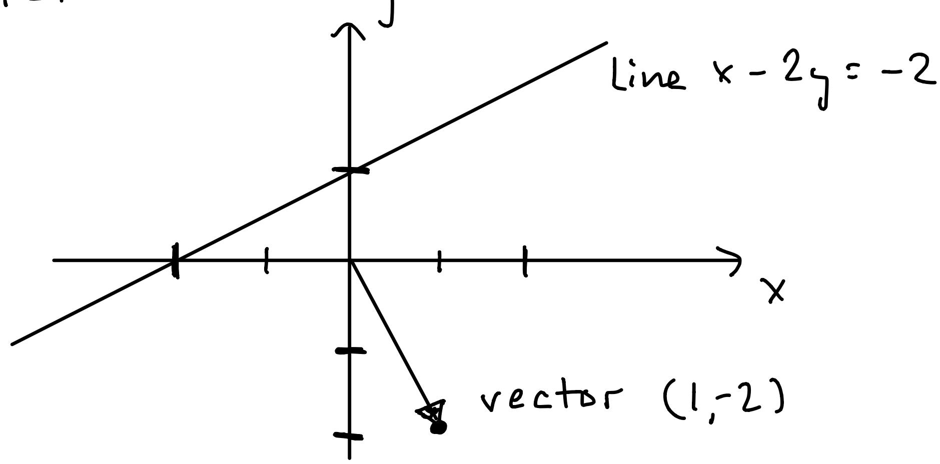
(a) $\vec{a} = (1,-2)$

$\vec{x} = (x,y)$

$\vec{a} \circ \vec{x} = -2$

$x - 2y = -2$.

Picture :



Note that the line $\vec{a} \circ \vec{x} = -2$ is perpendicular to the vector $\vec{a}$.

(b) In general I claim that the line $\vec{a} \circ \vec{x} = c$ is $\perp$ to vector $\vec{a}$.

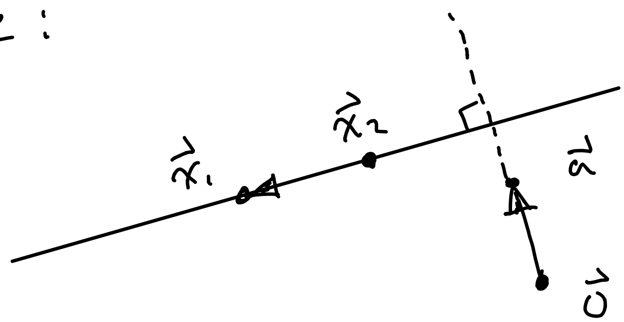
<u>Proof</u>: If points $\vec{x}_1$ & $\vec{x}_2$ are on the line then $\vec{a} \circ \vec{x}_1 = c$ & $\vec{a} \circ \vec{x}_2 = c$,

and it follows that

$$\vec{a} \cdot (\vec{x}_1 - \vec{x}_2) = \vec{a} \cdot \vec{x}_1 - \vec{a} \cdot \vec{x}_2$$
$$= c - c$$
$$= 0,$$

hence $\vec{a}$ is $\perp$ to $\vec{x}_1 - \vec{x}_2$.

Picture:



(c) I claim that point $x = \dfrac{c}{\|\vec{a}\|^2}\vec{a}$

is on the line $\vec{a} \cdot \vec{x} = c$.

Proof:

$$\vec{a} \cdot \left(\frac{c}{\|\vec{a}\|^2} \vec{a} \right)$$

$$= \frac{c}{\|\vec{a}\|^2} (\vec{a} \cdot \vec{a}) = \frac{c}{\cancel{\|\vec{a}\|^2}} \cancel{\|\vec{a}\|^2} = c \checkmark$$

See the notes for discussion.