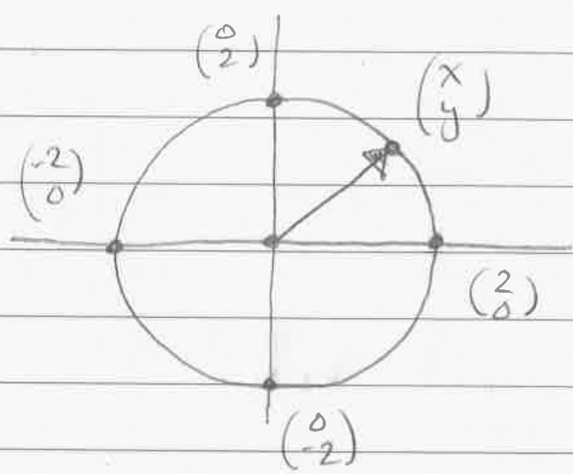
May 29 - June 2

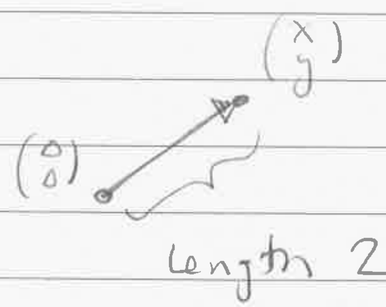
New Topic : Systems of Equations.

We have seen how Cartesian allow us to replace the geometric notions of "length" and "angle" with the algebra of "vectors" and "dot product".

Next we will discuss how to replace "geometric shapes" by "equations".

Example : Consider the circle of radius 2 centered at the origin in the Cartesian plane.





If $\begin{pmatrix} x \\ y \end{pmatrix}$ is any point on the circle then we must have

$$\left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\| = 2$$

$$\left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\|^2 = 4$$

$$\boxed{x^2 + y^2 = 4}$$

We say that this is the <u>equation</u> of the circle. In other words, the circle consists of all points $\begin{pmatrix} x \\ y \end{pmatrix}$ such that $x^2 + y^2 = 4$.

Example: What shape does the equation

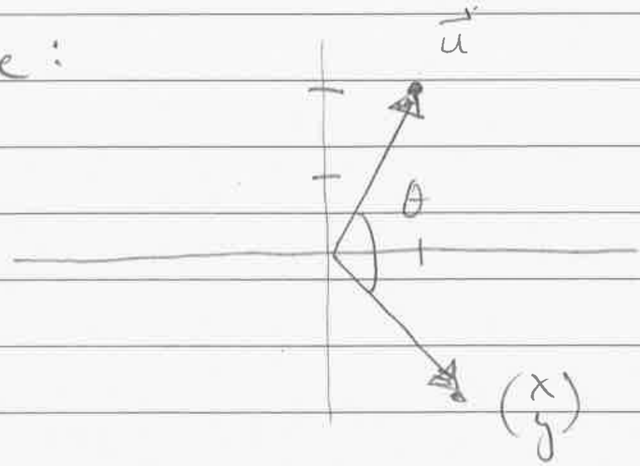
$$x + 2y = 0 \text{ represent?}$$

Here's a cool trick: Consider the vector $\vec{u} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Then we can express the quantity $x + 2y$ as the dot product of $\vec{u}$ with some "variable vector"

$$x + 2y = \begin{pmatrix} x \\ y \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\| \cdot \left\| \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\| \cos\theta .$$

$$= \sqrt{x^2 + y^2} \cdot \sqrt{5} \cdot \cos\theta$$

Picture:



The equation $x+2y=0$ is now the same as the equation

$$\sqrt{x^2+y^2}\cdot\sqrt{3}\cdot\cos\theta = 0,$$

which is true if and only if $\cos\theta = 0$ (and hence $\theta = 90^\circ$ or 270°).

[Jargon: Given vectors $\vec{u}$ & $\vec{v}$ we say that $\vec{u}$ & $\vec{v}$ are perpendicular (or orthogonal) precisely when $\vec{u}\cdot\vec{v}=0$. In this case we write

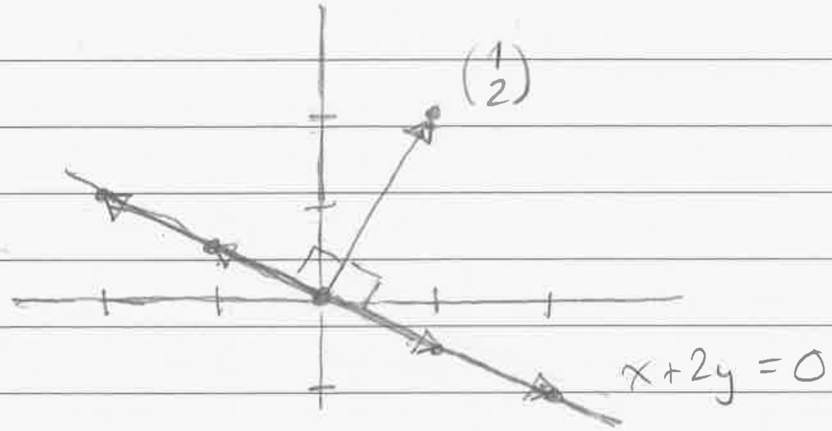
$$\vec{u}\perp\vec{v}.$$

]

So the equation $x+2y=0$ corresponds to the set of points $\binom{x}{y}$ such that

$$\binom{x}{y}\perp\binom{1}{2}.$$

Picture:



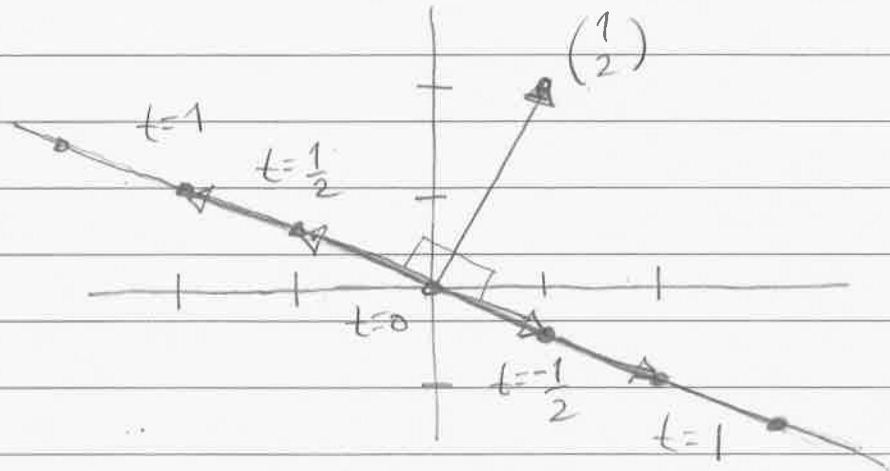
This is the unique line perpendicular to the vector $\begin{pmatrix}1\\2\end{pmatrix}$ and containing the point $\begin{pmatrix}0\\0\end{pmatrix}$.

Alternatively, we could express this line in "parametric form". To do this we let $y=t$ be a "free parameter" and solve for x & y in terms of t:

$$x+2y=0$$
$$x=-2y=-2t$$

$$\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}-2t\\t\end{pmatrix}=\begin{pmatrix}-2t\\1t\end{pmatrix}=t\begin{pmatrix}-2\\1\end{pmatrix}.$$

We obtain all multiples of the vector $\begin{pmatrix}-2\\1\end{pmatrix}$.
Let's add this to our picture.



So we can think of this as the "line perpendicular to $\binom{1}{2}$" or the "line in the direction of $\binom{-2}{1}$".

$\boxed{x + 2y = 0}$ $\longleftrightarrow$ $\boxed{\begin{pmatrix} x \\ y \end{pmatrix} = t\begin{pmatrix} -2 \\ 1 \end{pmatrix}}$

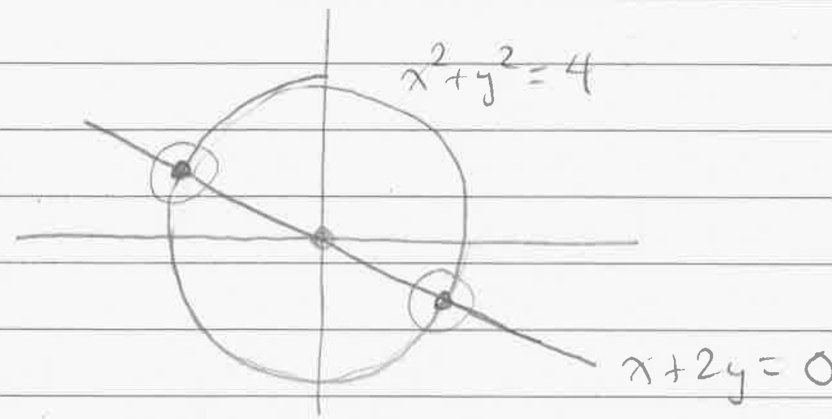
same thing

[Unfortunately, neither of these representations for the line is unique. :( ]

Example: Now, try to find all points $\binom{x}{y}$ such that the following two equations hold SIMULTANEOUSLY!

$$\begin{cases} x^2 + y^2 = 4 \\ x + 2y = 0 \end{cases}$$

Picture



Solving simultaneous equations is the same thing as finding the intersection of geometric shapes. In this case we expect exactly two solutions

How to find them? For convenience, let's name the equations.

$$\text{(1)} \quad x^2+y^2 = 4$$
$$\text{(2)} \quad x+2y = 0.$$

Solve (2) for x,

$$x+2y = 0$$
$$x = -2y$$

and then substitute this value into (1),

$$x^2 + y^2 = 4$$
$$(-2y)^2 + y^2 = 4$$
$$4y^2 + y^2 = 4$$
$$5y^2 = 4$$
$$y^2 = 4/5$$

$$y = \pm 2/\sqrt{5}.$$

We have successfully "eliminated" the variable x. Now we substitute these two values of y back into ② ($x = -2y$) to get the two solutions

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4/\sqrt{5} \\ -2/\sqrt{5} \end{pmatrix} \text{ or } \begin{pmatrix} -4/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}$$

$$= \frac{-2}{\sqrt{5}}\begin{pmatrix} -2 \\ 1 \end{pmatrix} \text{ or } \frac{2}{\sqrt{5}}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

[$t = -2/\sqrt{5}$ or $2/\sqrt{5}$ in our old parametrization of the Line]

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In summary, we can think of an equation in n variables as a certain "shape" in n-dimensional space. Solving equations simultaneously means finding the intersection of the corresponding shapes.

Again we have a bridge between algebra and geometry:

Algebra $\longleftrightarrow$ Geometry

(simultaneous equations) (intersections of shapes)

And, again, we will see that each side enlightens the other.

Last time we discussed the idea than an equation represents a shape in space.

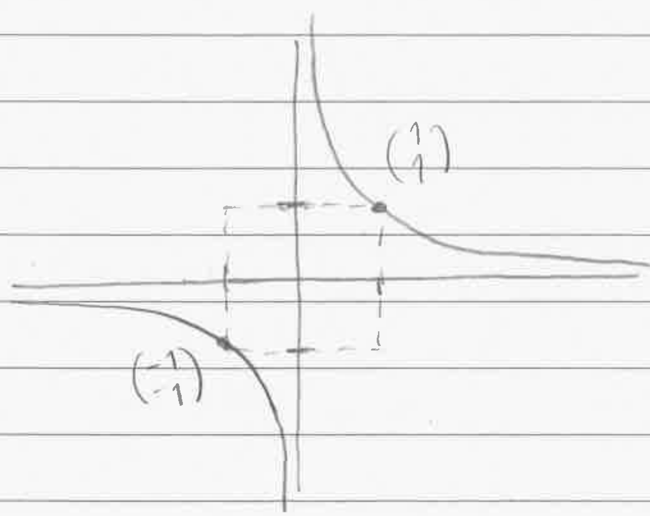
But what kinds of shapes can be represented this way?

Examples:

- $x^2 + y^2 = 4$ is a circle in 2D
- $x + 2y = 0$ is a line in 2D.

What shape is represented by $xy = 1$?

We can solve for y to get $y = 1/x$.



This shape is called a "hyperbola".

In general we see that <u>one</u> equation in <u>two</u> unknowns represents some kind of "one-dimensional curve" in the Cartesian plane.

3D Example: What shape is represented by the equation

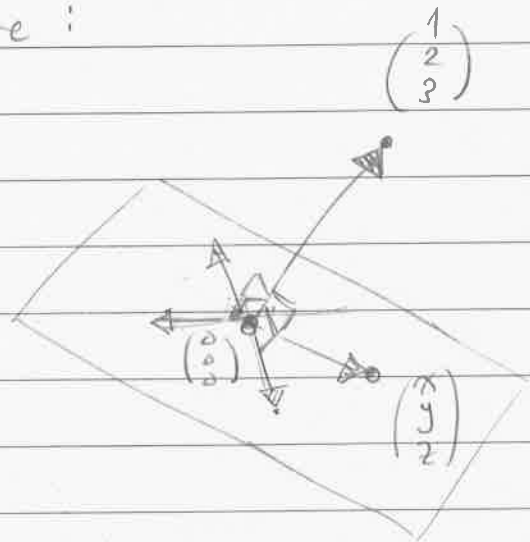
$$x + 2y + 3z = 0 \quad ?$$

Let's use the same trick from Monday. We will rewrite the equation using the dot product:

$$x + 2y + 3z = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0.$$

This equation says that the vectors $(1,2,3)$ and (x,y,z) are <u>perpendicular</u>. The collection of all such (x,y,z) form a <u>plane</u> in three-dimensional space.

Picture :



Every vector in the plane $x + 2y + 3z = 0$ is perpendicular to the vector $(1,2,3)$.

We can describe this as the unique plane that is perpendicular to the vector $(1,2,3)$ and contains the point $(0,0,0)$.

We can also describe this plane "parametrically", but since a plane is a "two-dimensional" shape we will require two "free parameters".

Let's say $y = s$ & $z = t$ are free (the choice is arbitrary). Then we can solve for x, y, z in terms of s & t. We have

$$\begin{aligned} x + 2y + 3z &= 0 \\ x &= -2y - 3z \\ &= -2s - 3t \end{aligned}$$

and hence

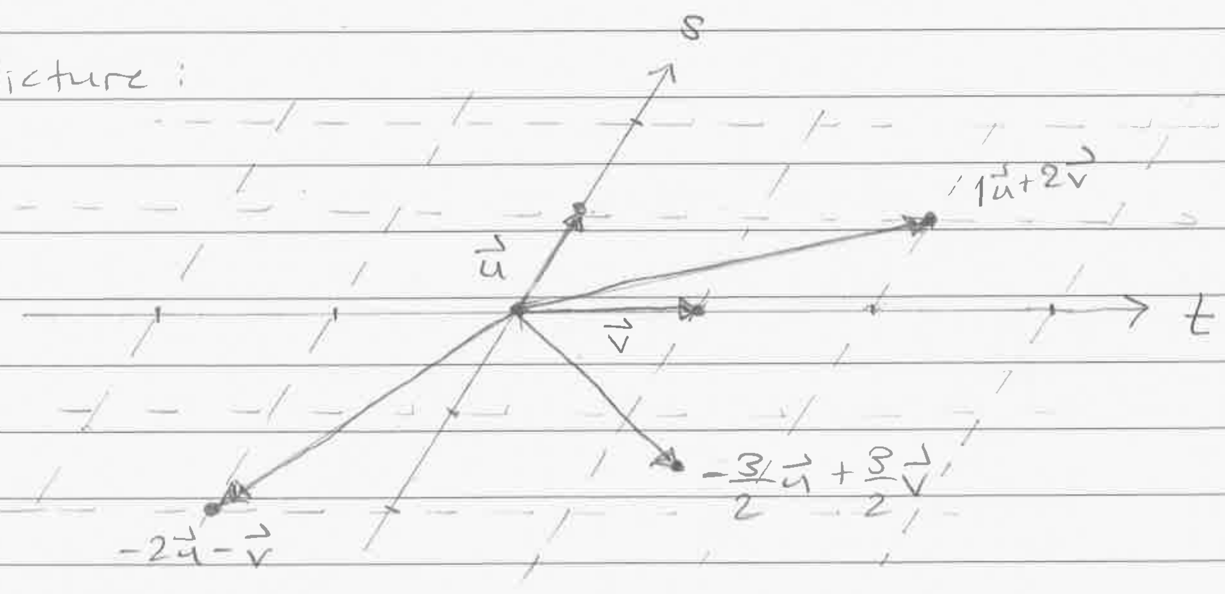
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2s-3t \\ s \\ t \end{pmatrix} \overset{\text{TRICK}}{=} \begin{pmatrix} 0-2s-3t \\ 0+1s \quad 0t \\ 0+0s+1t \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$$

$$\boxed{\begin{pmatrix} x \\ y \\ z \end{pmatrix} = s\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}}$$

We have successfully "parametrized" the plane. To put this another way, define the vectors $\vec{u} = (-2,1,0)$ & $\vec{v} = (-3,0,1)$. We say that our plane is <u>spanned</u> (or <u>generated</u>) by the vectors $\vec{u}$ & $\vec{v}$.

Picture :



Every point in the plane can be <u>uniquely</u> represented in the form $s\vec{u} + t\vec{v}$, Thus we have defined a "coordinate system" on the plane. If we want to be really bold we might even use the notation

$$s\vec{u} + t\vec{v} = \text{"}\begin{pmatrix} s \\ t \end{pmatrix}\text{"}$$

for points in the plane.

Thinking Problem: Why can't we just use the "standard" coordinate system for this plane?

Solution: Because it doesn't <u>have</u> a standard coordinate system. We had to make one up from scratch. There are infinitely many ways to do it and there is no "best" way.

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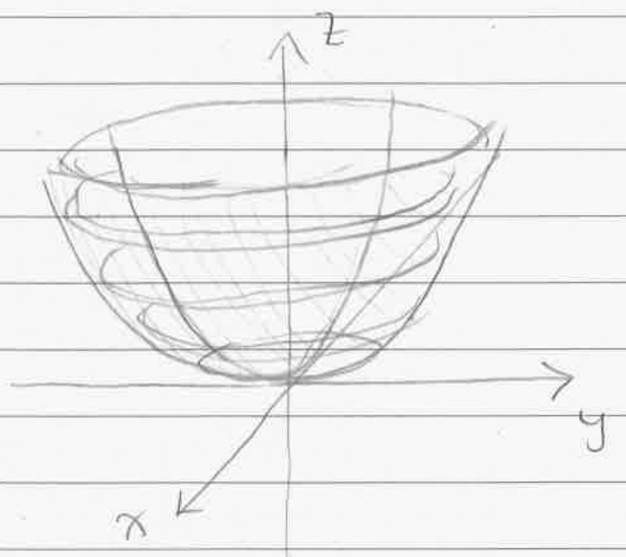
Another 3D Example: Consider the equation

$$x^2 + y^2 = z.$$

For a fixed value of z, say $z = a$, we obtain the circle $x^2 + y^2 = a$ of radius $\sqrt{a}$.

When $x = 0$ we obtain $y^2 = z$, a parabola in the y,z-plane. When $y = 0$ we obtain $x^2 = z$, a parabola in the x,z-plane.

Putting these "cross-sections" together gives the following picture:



It's a hollow bowl-shaped surface called a "paraboloid"

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From these examples we observe:

One equation in three unknows represents a "two-dimensional surface" living in "three-dimensional space".

1/29/16

HW2 due next Wed Feb 3.

Last time we made the following observations.

- 1 equation in 2 unknowns represents a "1D curve" living in 2D space

Examples: lines, circles, parabolas, hyperbolas, ellipses, etc.

- 1 equation in 3 unknowns represents a "2D surface" living in 3D space.

Examples: planes, spheres, paraboloids, ellipsoids, hyperboloids, etc.

OK, but what about "1D curves" living in 3D space? For example, what is the equation of a line in 3D?

Answer: There is no such thing as

"the equation" of a line in 3D!

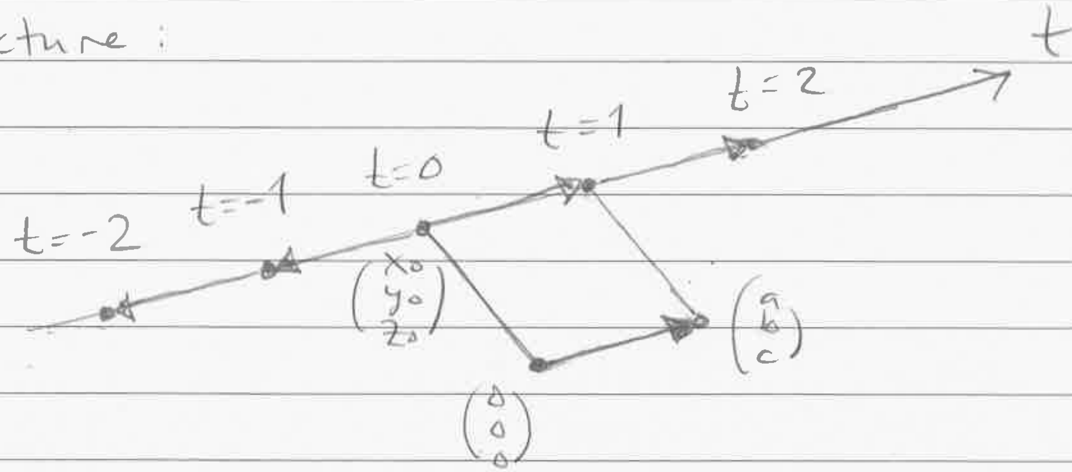
OK, so how can we describe a line in 3D?

There are two ways.

1. We can give a "parametrization" of the line in the following form:

$$(*) \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + t \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

Picture:



This is the unique line in the direction of the vector (a,b,c) and containing the point (x_0, y_0, z_0).

[Maybe we think of (x,y,z) as the position of a particle at "time t". Then (x_0, y_0, z_0) is the "initial position", i.e., at time $t=0$.]

Objection: Couldn't we call (*) "the equation" of the line? NO, for two reasons. First, because it contains the parameter t. Second, because the "vector equation" (*) really represents a system of three "number equations".

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 + ta \\ y_0 + tb \\ z_0 + tc \end{pmatrix}$$

is the same as

$$\begin{cases} x = x_0 + ta \\ y = y_0 + tb \\ z = z_0 + tc \end{cases}.$$

///

2. We can represent a line in 3D by a <u>system</u> of (at least) 2 equations in 3 unknowns.

For example, let's compute the <u>intersection</u> of the planes

$$x + y + z = 0 \quad \& \quad x + 2y + 3z = 0.$$

The points (x, y, z) in the intersection are the solutions to the system of simultaneous equations

$$\begin{cases} x + y + z = 0 & (1) \\ x + 2y + 3z = 0 & (2) \end{cases}$$

Now is a good time to introduce our fundamental tool for solving systems of equations.

☆ The Idea of Elimination:

- Given a bunch of true equations, we can produce more true equations by adding/subtracting equations and multiplying equations by numbers.

- We will try to produce true equations with fewer variables.

In the above system, we can produce an equation with no x by subtracting

$$(2) - (1)$$

$$\textcircled{2} \quad x + 2y + 3z = 0$$
$$\textcircled{1} \quad x + y + z = 0$$
$$\textcircled{2} - \textcircled{1} \quad 0x + 1y + 2z = 0$$

Let's call this new equation

$$\textcircled{3} \quad y + 2z = 0.$$

Can we eliminate y or z from this equation?
NO. We don't have enough equations to do that.

But we <u>can</u> eliminate y from equation ②
as follows

$$\textcircled{2} \quad x + 2y + 3z = 0$$
$$2\textcircled{3} \quad 2y + 4z = 0$$

$$\textcircled{2} - 2\textcircled{3} \quad x + 0y - z = 0.$$

Let's call this new equation

$$\textcircled{4} \quad x - z = 0.$$

What have we done?

We can now replace the system

$$\begin{cases} x + y + z = 0 & (1) \\ x + 2y + 3z = 0 & (2) \end{cases}$$

by the simpler system

$$\begin{cases} x \quad - z = 0 & (4) \\ \quad y + 2z = 0 & (3) \end{cases}$$

The good news is that the simpler system has the same solution, so we can just throw the old one away !!

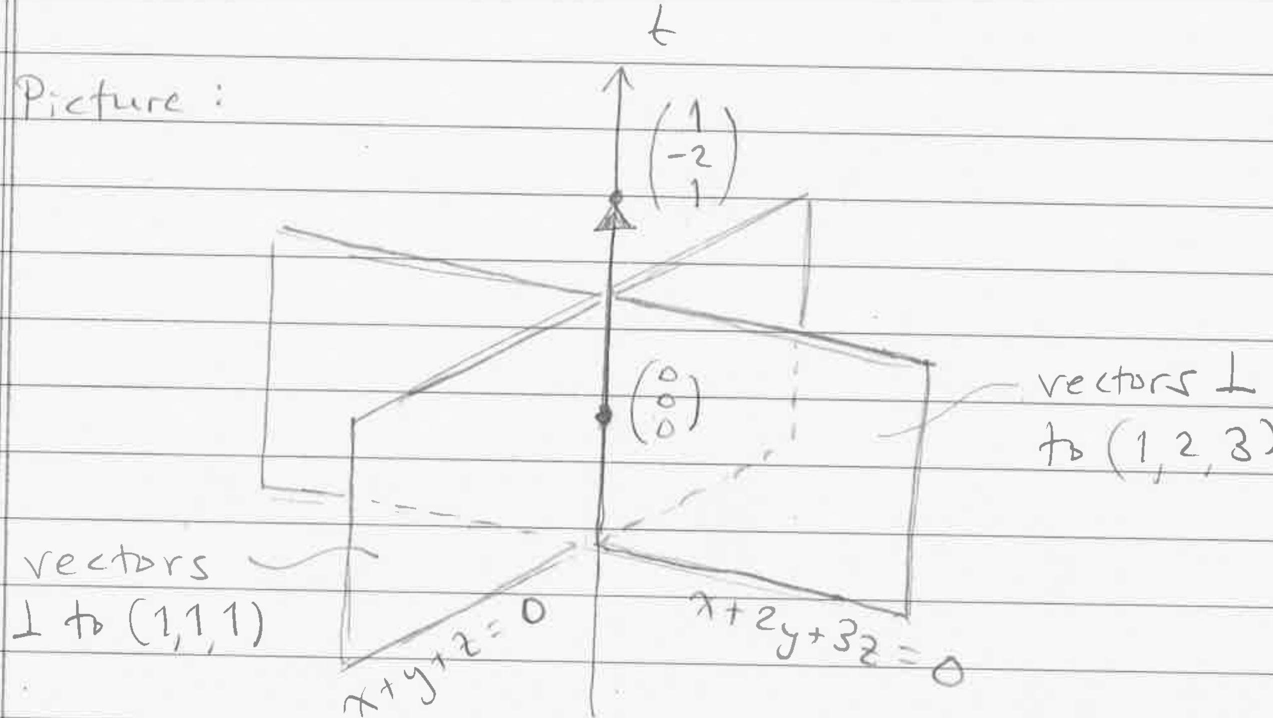
What is the solution. Well, we have basically solved for x & y in terms of the "free parameter" z.

Letting $z = t$ we can write.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ -2z \\ z \end{pmatrix} = z \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = t \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$

This is a parametrized line in 3D.

Picture :



The intersection of the planes

$$x + y + z = 0 \quad \& \quad x + 2y + 3z = 0$$

is the line $(x,y,z) = t(1,-2,1)$.

[Another interpretation : Since $x+y+z = 0$ is the plane perpendicular to $(1,1,1)$ and $x+2y+3z = 0$ is the plane perpendicular to $(1,2,3)$, the line $t(1,-2,1)$ consists of all vectors that are SIMULTANEOUSLY PERPENDICULAR to both $(1,1,1)$ & $(1,2,3)$.]

Last time we considered the following system of 2 equations in 3 unknowns:

$$\begin{cases} x + y + z = 0 \\ x + 2y + 3z = 0 \end{cases}$$

We used the method of "elimination" to reduce this to a simpler, but still equivalent, system:

$$\begin{cases} x \quad\quad - z = 0 \\ \quad\quad y + 2z = 0 \end{cases}$$

Then we could read off the solution:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ -2z \\ z \end{pmatrix} = z \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$

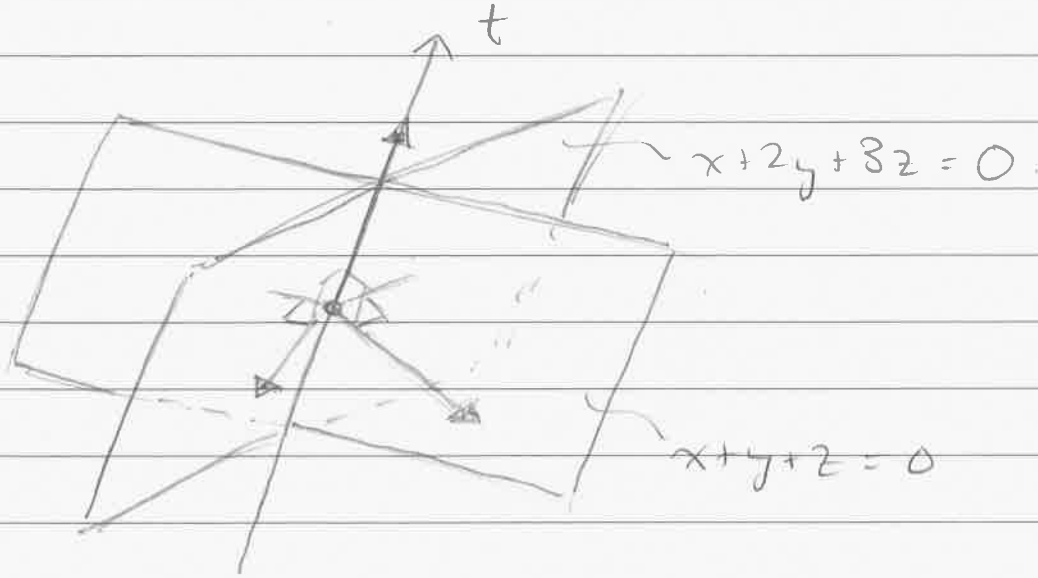
If we want, we can let $z = t$ be a "parameter". We a parametrized line living in 3D space:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = t \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$

There are two geometric ways to think of this solution.

1. The equations $x+y+z=0$ and $x+2y+3z=0$ represent planes in 3D and $(x,y,z) = t(1,-2,1)$ is their <u>line of intersection</u>.

Picture:

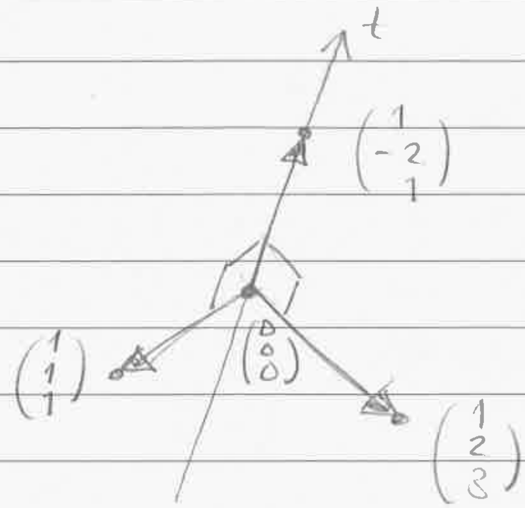


2. The equations

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \quad \& \quad \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

say that (x,y,z) is a vector that is simultaneously perpendicular to both $(1,1,1)$ & $(1,2,3)$. The set of all such vectors is $(x,y,z) = t(1,-2,1)$.

Picture:



[Remark: You might be familiar with the concept of the "cross product" from physics or vector calculus. Given two "three-dimensional" vectors $\vec{u} = (u_1, u_2, u_3)$ & $\vec{v} = (v_1, v_2, v_3)$, we define the vector

$$\vec{u} \times \vec{v} := \begin{pmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{pmatrix}$$

The purpose of this definition is that

$\vec{u} \times \vec{v}$ is a vector that is simultaneously perpendicular to both $\vec{u}$ & $\vec{v}$. In other words, the solution of the system

$$\begin{cases} u_1 x + u_2 y + u_3 z = 0 \\ v_1 x + v_2 y + v_3 z = 0 \end{cases}$$

is given by $(x, y, z) = t(\vec{u} \times \vec{v})$.

Example:

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1\cdot 3 - 1\cdot 2 \\ 1\cdot 1 - 1\cdot 3 \\ 1\cdot 2 - 1\cdot 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$

]

We have seen the following:

- 1 equation in 2 unknowns
  $\rightsquigarrow$ 1D curve in 2D space

- 1 equation in 3 unknowns
  $\rightsquigarrow$ 2D surface in 3D space

- 2 equations in 3 unknowns
  $\rightsquigarrow$ 1D curve in 3D space

I'll just ask you to believe the following statement:

☆ m equations in n unknowns (most likely) determine an "$n-m$ dimensional shape" living in "n-dimensional space".

Or, in other words,

☆ every new equation (probably) decreases the dimension of the solution by 1.

I'll also let you in on a little secret:

☆ Solving a general system of m equations in n unknowns is essentially impossible.

Therefore, in MTH 210 we will restrict our attention to systems of LINEAR equations. And what are these?

Definition: A Linear equation in n unknowns $x_1, x_2, \cdots, x_n$ has the form

(*) $\boxed{a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b}$

for some numbers $a_1, a_2, \ldots, a_n$ and b. ///

Q: What kind of geometric shape is represented by the equation (*) ?

A: We've seen that

$n=2 \rightsquigarrow$ (*) is a line living in 2D
$n=3 \rightsquigarrow$ (*) is a plane living in 3D.

So in general we expect that (*) represents some kind of "<u>flat</u> $(n-1)$-dimensional shape" living in "n-dimensional space". And we have a name for this kind of shape:

it's called a <u>hyperplane</u>.

In general we will use the term <u>d-plane</u> to refer to any "flat d-dimensional shape".

So, inside n-dimensional space we have the following kinds of flat shapes:

0-planes = points
1-planes = lines
2-planes = planes
⋮
(n-1)-planes = hyperplanes
n-planes = the full space.

The intersection of flat shapes is always flat, so we can make the following statement:

A system of m LINEAR equations in n unknowns represents the intersection of m "hyperplanes" living in "n-dimensional space".

☆

The intersection is most likely an "(n-m)-plane", but it might possibly be a "d-plane" for some other d if the original m hyperplanes were not in "general position".

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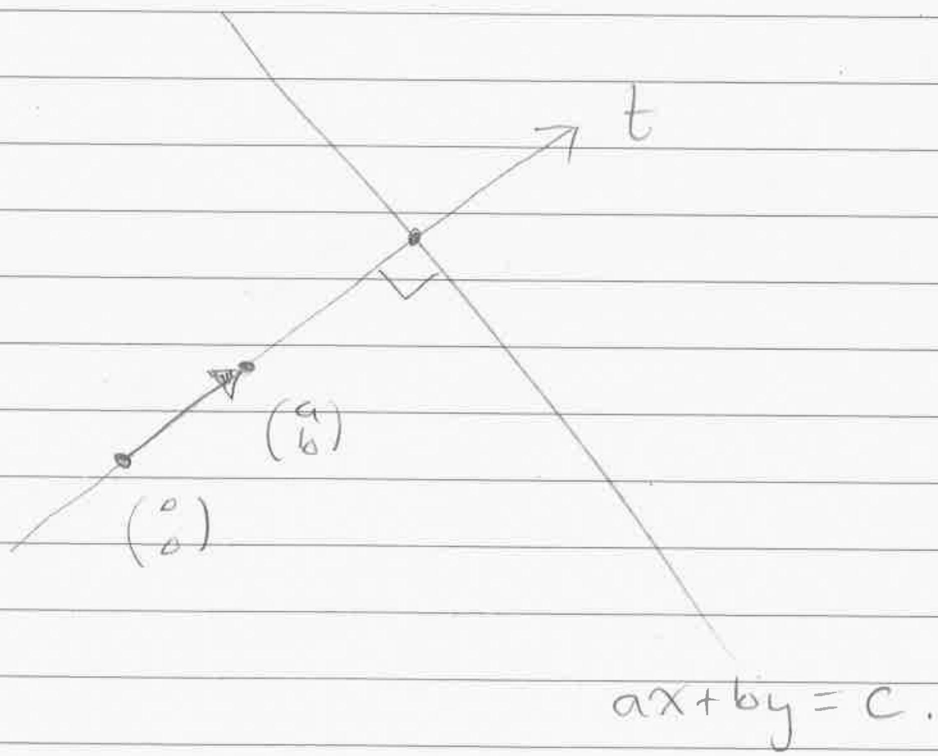
Today: HW 2 Discussion.

Problem 1′: Let a, b & c be constants and let x, y be variables [this convention goes back to Descartes]. The line

$$ax + by = c$$

is perpendicular to the vector (a, b).

Picture:



There are infinitely many points on the line. Are any of them special?

Well, there is one point on the line that is closest to $(0,0)$. I think that's pretty special. For geometric reasons, this point is the intersection of the line $ax+by=c$ with the perpendicular line $(x,y)=t(a,b)=(ta,tb)$.

Let's compute it. We have a system of 3 equations in 3 unknowns x,y,t:

$$\begin{cases} x = ta \\ y = tb \\ ax+by=c. \end{cases}$$

But it's pretty easy to solve so we don't need any fancy technique. Just substitute:

$$\begin{aligned} ax+by &= c \\ a(ta)+b(tb) &= c \\ ta^2+tb^2 &= c \\ t(a^2+b^2) &= c \\ t &= c/(a^2+b^2). \end{aligned}$$

The point of intersection is

$$\begin{pmatrix} x \\ y \end{pmatrix} = t \begin{pmatrix} a \\ b \end{pmatrix} = \frac{c}{a^2+b^2} \begin{pmatrix} a \\ b \end{pmatrix}.$$

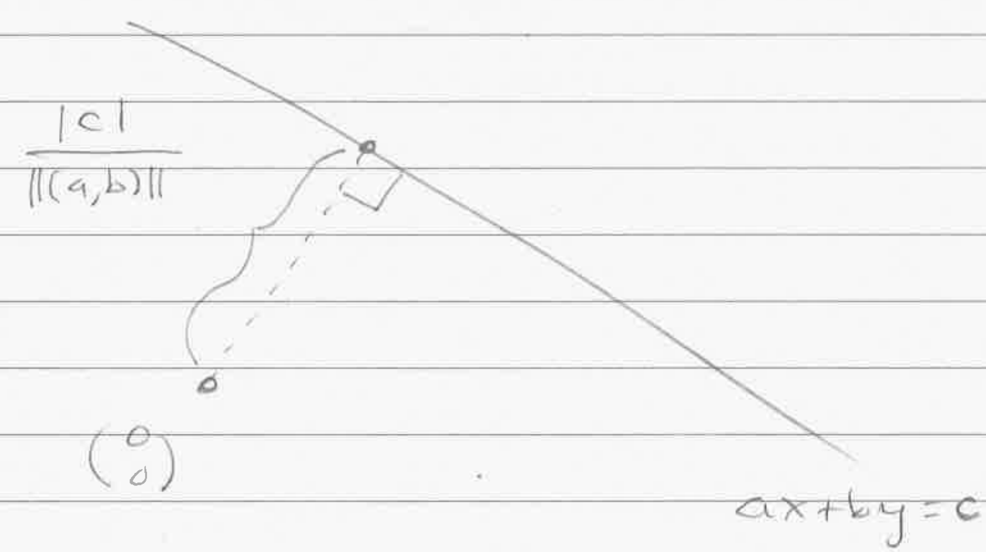
So this is the <u>closest</u> point on the line $ax+by=c$ to the origin.

HOW close is it?

After a bit of calculation (omitted for todays discussion) you will find that the distance between $(0,0)$ & $\frac{c}{a^2+b^2}(a,b)$ is

$$|c| \,/\, \|(a,b)\|.$$

Picture:



Later in the course we will be all about computing the minimum distance from a certain point to a certain "d-plane".

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Problem 3': The single vector equation

$$(*) \quad x\begin{pmatrix}-1\\1\end{pmatrix} + y\begin{pmatrix}2\\0\end{pmatrix} = \begin{pmatrix}3\\2\end{pmatrix}$$

is equivalent to the system of two simultaneous number equations

$$(**) \quad \begin{cases} -x + 2y = 3 \\ x = 2. \end{cases}$$

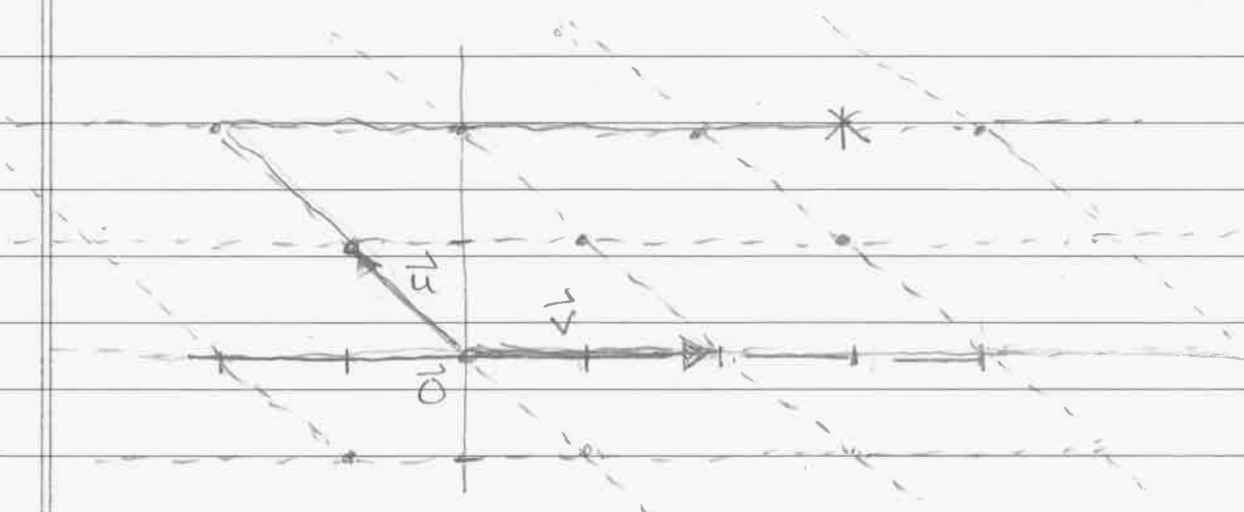
Therefore they have the same solution, which happens to be

$$\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}2\\5/2\end{pmatrix} \quad \text{or} \quad \begin{cases} x = 2 \\ y = 5/2. \end{cases}$$

But our mental pictures of the problems (*) and (**) are quite different.

In (*) we have a "target vector" $(3,2)$ and we want to get there but we are only allowed to travel in the directions $\vec{u} := (-1,1)$ & $\vec{v} := (2,0)$.

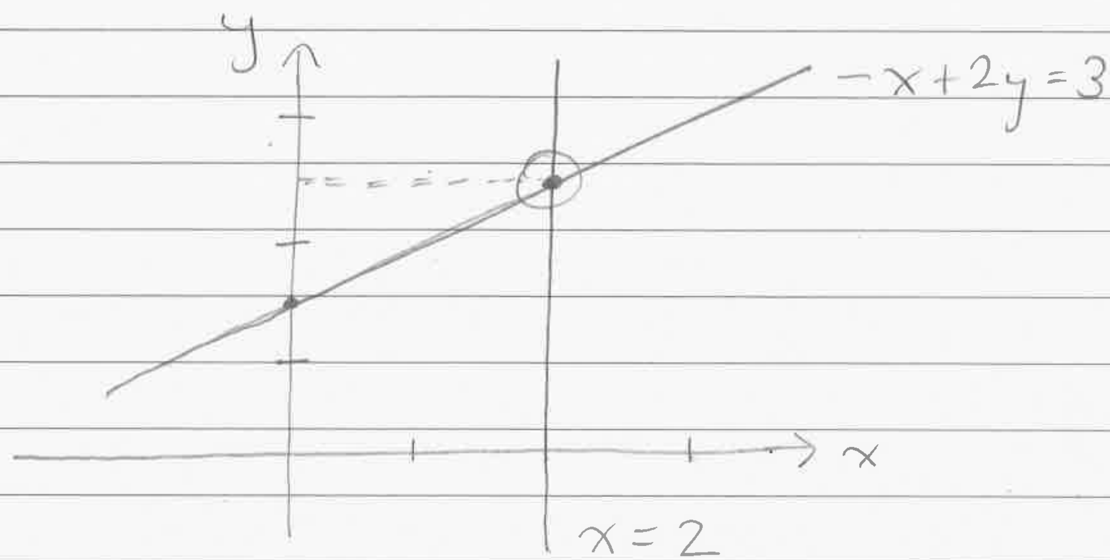
Let's think of $\vec{u}$ & $\vec{v}$ as a new "coordinate system" for the plane:



We start at the point $\vec{0}$. To get to the restaurant at $(3,2)$ we must travel <u>2 blocks</u> in the $\vec{u}$ direction and <u>5/2 blocks</u> in the $\vec{v}$ direction.

Gilbert Strang calls this the "column picture" of the system.

In (**) we have two equations representing lines in the plane. The solution of the system is interpreted as the point of intersection. So let's draw the lines:



Note that $-x + 2y = 3$ is the same as $y = 1/2\ x + 3/2$ in "slope/y-intercept" form.

The point of intersection is $(x, y) = (2, 5/2)$.

Gilbert Strang calls this the "row picture" of the system.

The point I want to emphasize is that the "row" & "column" pictures are just two different visualizations for the

SAME MATHEMATICAL PROBLEM.

In general it is a strength when we have multiple ways to visualize a mathematical problem because it gives us more opportunities to use our intuition.

So far we have been building intuition and learning the background material. Today, we are finally ready to state the central problem of linear algebra.

Recall that a system of m equations in n unknowns (most likely) represents an $(n-m)$-dimensional shape living in n-dimensional space. [That is, each new equation probably reduces the dimension of the solution by 1.]

Unfortunately, the problem of general equations is mostly impossible so in this course we will focus on a very special kind of equations.

☆ Definition: A linear equation in the n unknowns $x_1, x_2, \ldots, x_n$ has the form

$$\boxed{a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b}$$

where $a_1, a_2, \dots, a_n$ and b are constants.
We can also express this equation as

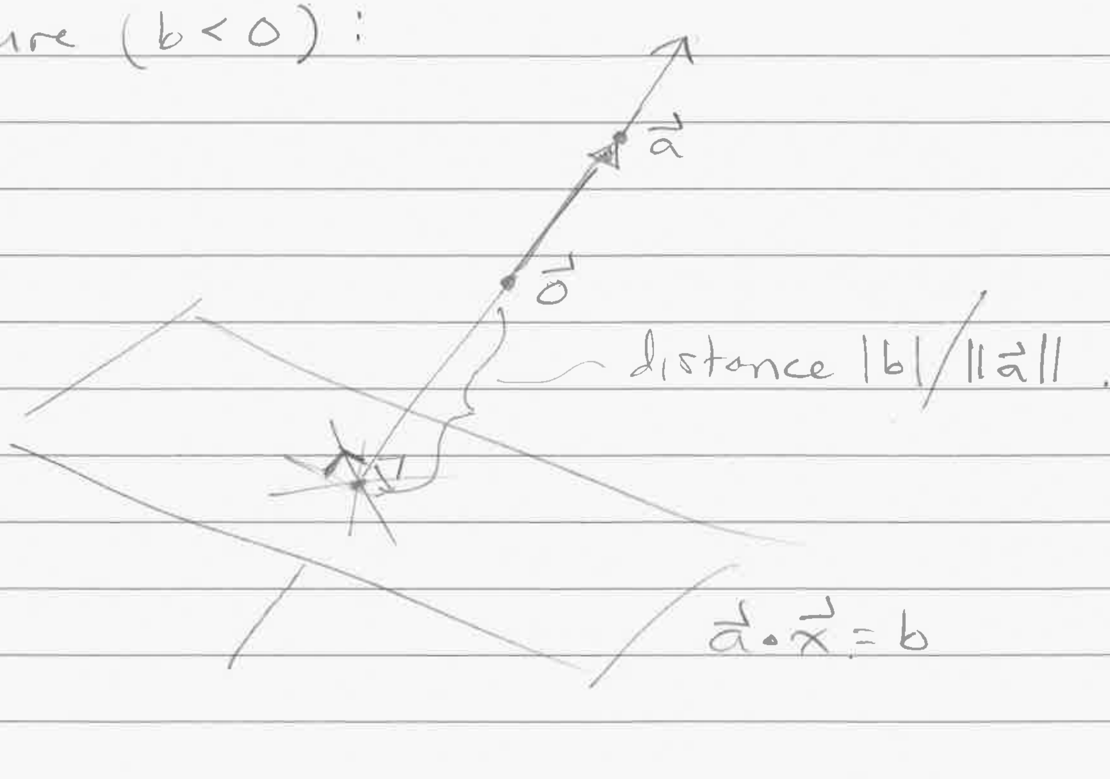
$$\boxed{\vec{a} \cdot \vec{x} = b}$$

where $\vec{a} = (a_1, a_2, \dots, a_n)$ & $\vec{x} = (x_1, \dots, x_n)$.

Geometrically, this is the "hyperplane" perpendicular to $\vec{a}$ that has distance $|b| / \|\vec{a}\|$ from the origin

[Recall that a "hyperplane" is an $(n-1)$-dimensional flat shape living in n-dimensional space.]

Picture $(b < 0)$:



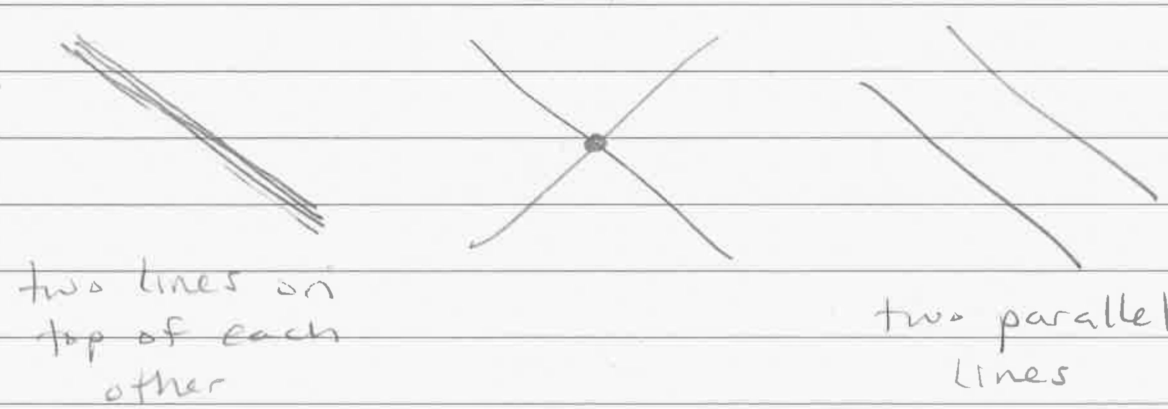
Special Case: If $b = 0$ then the hyperplane contains the point $\vec{0}$; otherwise not.

Finally, here it is:

☆ The Central Problem of Linear Algebra is to solve a system of m linear equations in n unknowns.

Geometrically, this means we want to compute the intersection of m hyperplanes living in n-dimensional space. We know that the answer will be a "d-plane" [flat d-dimensional shape] for some d. Probably we will have $d = n - m$, but funny things can happen sometimes.

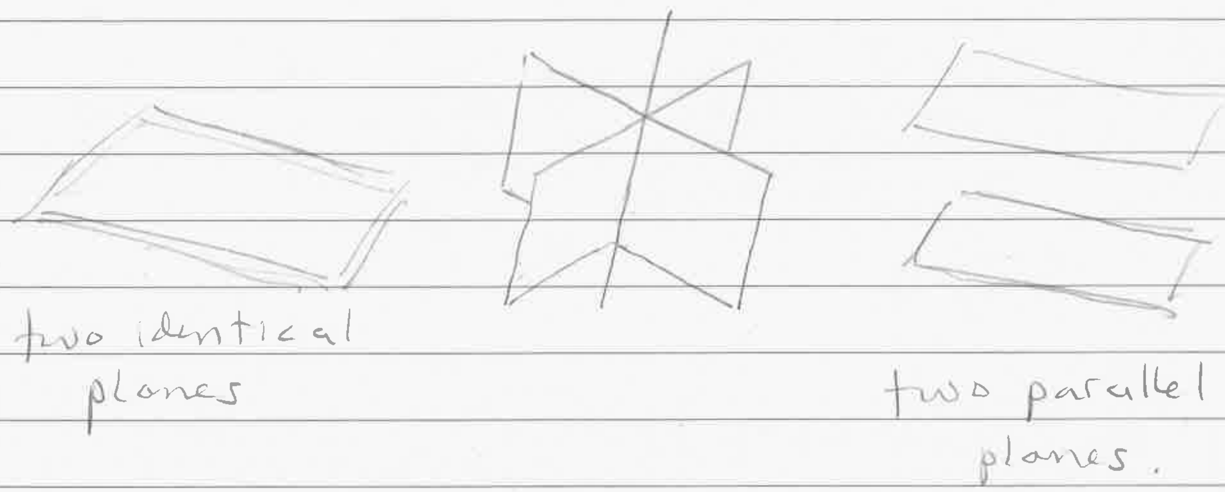
Example ($m = 2$, $n = 2$): The intersection of two lines can be a line (1-plane), a point (0-plane) or empty.



Intersecting at a point is most likely.

[Remark: Some people call an empty intersection a "(-1)-plane"! You do not need to call it that.]

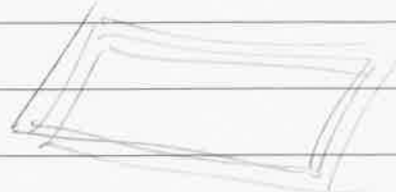
Example ($m=2, n=3$): Two planes in 3D can intersect in a plane (2-plane), a line (1-plane) or have empty intersection.



The line is most likely.

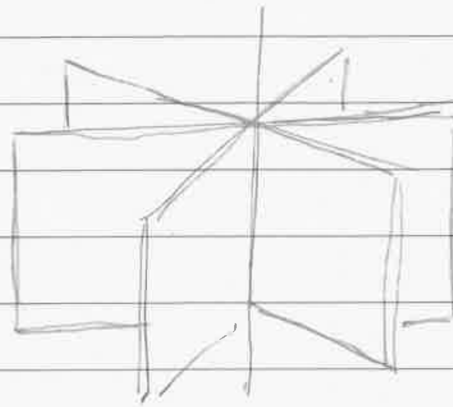
Example ($m=3, n=3$): Three planes in 3D can intersect in a plane (2-plane), a line (1-plane), a point (0-plane); or have empty intersection.

2-plane

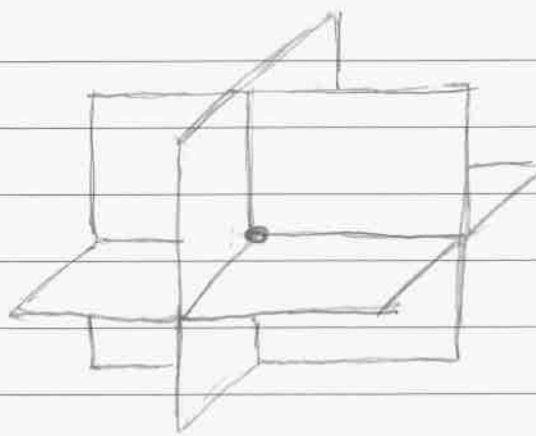


three identical planes

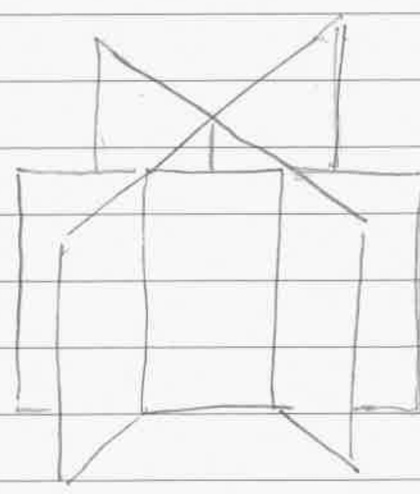
1-plane



0-plane



empty



Intersecting in a point is most likely.

In higher dimensions the number of possible configurations explodes and the pictures are impossible to draw. Nevertheless, humans have a complete and satisfactory way to solve this problem.

Q: How do they do it?

A: With algebra!

Example: Solve the linear system

$$\begin{cases} x + y + z = 2 & (1) \\ x + 2y + z = 3 & (2) \\ 2x + 3y + 2z = 5 & (3) \end{cases}$$

We will use the method of elimination. First eliminate x from (2) & (3) using (1):

$$\begin{array}{ll} (2) & x + 2y + z = 3 \\ (1) & x + y + z = 2 \end{array}$$

$$(2)-(1) \qquad y = 1 \qquad (2)'$$

Hey, that was lucky; z went away too!

$$\begin{array}{lrcl} ③ & 2x + 3y + 2z & = & 5 \\ ① & x + y + z & = & 2 \end{array}$$

$$\begin{array}{lrcll} ③ - 2① & y & = & 1 & ③' \end{array}$$

Our new equivalent system is

$$\begin{cases} x + y + z = 2 & ① \\ y = 1 & ②' \\ y = 1 & ③' \end{cases}$$

Next we eliminate y from ③$'$ using ②$'$:

$$\begin{array}{lrcl} ③' & y & = & 1 \\ ②' & y & = & 1 \end{array}$$

$$\begin{array}{lrcll} ③' - ②' & 0 & = & 0. & ③'' \end{array}$$

Oops, we got the true (but uninteresting) equation "$0 = 0$". Our new system is

$$\begin{cases} x + y + z = 2 & ① \\ y = 1 & ②' \\ 0 = 0 & ③'' \end{cases}$$

Finally we eliminate y from ① using ②$'$:

$$\begin{array}{lrcl} ① & x + y + z & = & 2 \\ ②' & y & = & 1 \end{array}$$

$$①-②' \quad x + z = 1 \qquad ①'$$

Our final equivalent system is

$$\left\{\begin{array}{rcll} x + z & = & 1 & ①' \\ y & = & 1 & ②' \\ 0 & = & 0 & ③'' \end{array}\right.$$

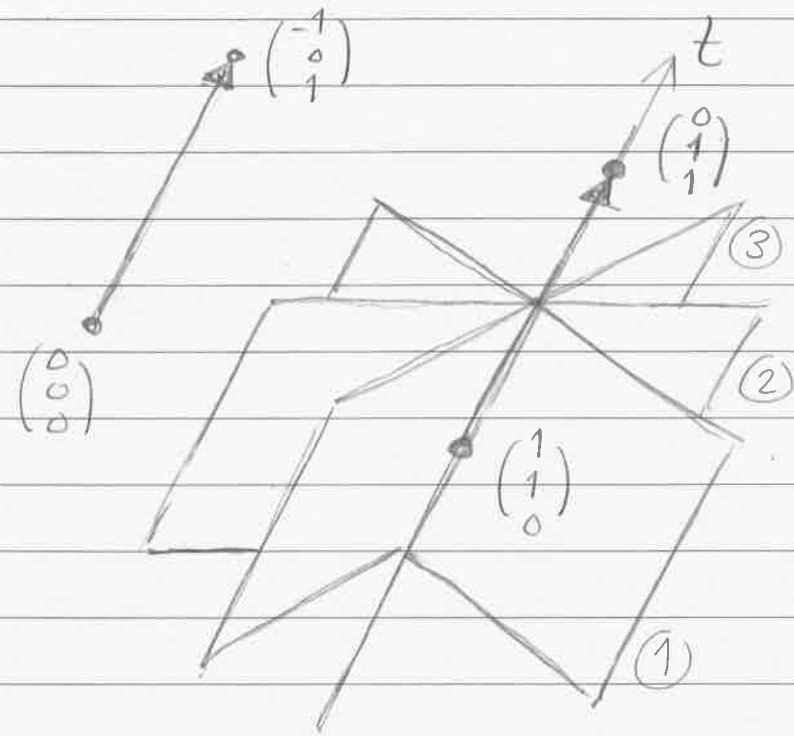
There is nothing left to do but read off the answer. We use $z = t$ as a parameter to get

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 - z \\ 1 \\ z \end{pmatrix} = \begin{pmatrix} 1 - 1z \\ 1 + 0z \\ 0 + 1z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

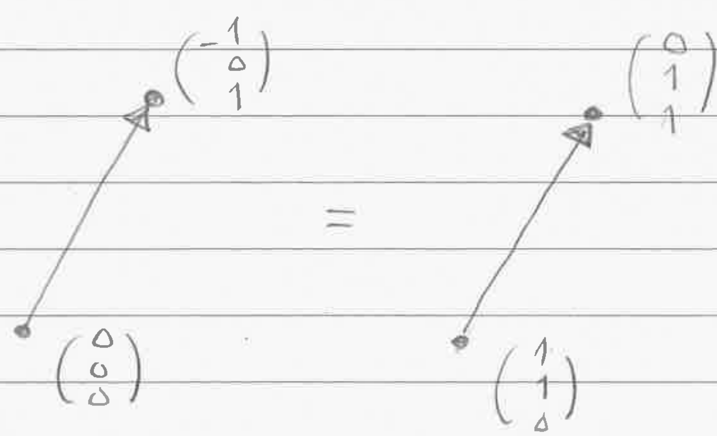
$$\boxed{\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}$$

This is a line.

Picture: The three original planes ①, ②, ③ meet in the line that contains the point (1,1,0) and is parallel to the vector (−1, 0, 1).



[Note that



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