Math 210 **Summer 2017**
Homework 6 **Drew Armstrong**

Let A be a square matrix. A scalar λ is called an *eigenvalue of A* if there exists a nonzero vector $\vec{v} \neq \vec{0}$ satisfying

$$\boxed{A\vec{v} = \lambda\vec{v}.}$$

Any such vector $\vec{v}$ is called a λ-*eigenvector*. If A is $n \times n$ then it has at most n different eigenvalues. They are the roots of the *characteristic equation*:

$$\boxed{\det(A - \lambda I) = 0.}$$

Problem 1. Suppose that λ is an eigenvalue of A and suppose that $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ are λ-eigenvectors. Show that any linear combination of the $\vec{v}_i$ is also a λ-eigenvector. [Conclusion: The collection of λ-eigenvectors for A forms a subspace called the λ-*eigenspace* of A.]

Problem 2. Consider the matrix $A = \begin{pmatrix} 5 & -2 \\ 12 & -5 \end{pmatrix}$.

(a) Write down the characteristic equation of A and solve it to find the eigenvalues. There will be two of them.
(b) For each of the two eigenvalues, find all of the corresponding eigenvectors.
(c) Draw the two eigenspaces of A, labeled by their eigenvalues.

Problem 3. Let $\vec{a}$ be any nonzero vector and consider the matrix $P = \vec{a}\vec{a}^T/\vec{a}^T\vec{a}$ that projects onto the line $t\vec{a}$.

(a) Show that $\vec{a}$ is an eigenvector of P. What is the corresponding eigenvalue?
(b) Let $\vec{b}$ be any vector perpendicular to $\vec{a}$. Show that $\vec{b}$ is an eigenvector of P. What is the corresponding eigenvalue?
(c) Draw the two eigenspaces of P, labeled by their eigenvalues.

Problem 4. Consider the same matrix P from Problem 3. I claim that the matrix $R = I - 2P$ performs a *reflection* across the (hyper)plane with normal vector $\vec{a}$. [You can just believe me about that.]

(a) If $\vec{v}$ is an eigenvector of P with eigenvalue λ, show that $\vec{v}$ is also an eigenvector of R, but with eigenvalue $1 - 2\lambda$. [Hint: Don't think.]
(b) Use this observation to find all of the eigenvalues and eigenvectors of R. [Hint: Don't do any work.]
(c) Draw the eigenspaces of R, labeled by their eigenvalues.