**Math 210 F** **Spring 2016**
**Homework 3** **Drew Armstrong**

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**Problem 1.** In class I stated that a system of **linear** equations has either 0, 1, or $\infty$ many solutions. Let's examine this claim.

(a) Suppose that $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$ are two solutions to the linear equation
$$ax + by + cz = d.$$
In this case, show that the midpoint $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$ is also a solution.

(b) Fill in the blank: If 25 hyperplanes in 12-dimensional space meet at two given points, they they must also meet at _______ .

**Problem 2.** Consider the following linear system:
$$\begin{cases} x + \phantom{2}y + \phantom{2}z = 2 \\ x + 2y + \phantom{2}z = 3 \\ x + 3y + 2z = 5 \end{cases}$$

(a) Compute the RREF of the system.
(b) Describe the row picture of the solution.
(c) Describe the column picture of the solution.

**Problem 3.** Now consider the modified system:
$$\begin{cases} \phantom{2}x + \phantom{2}y + \phantom{2}z = 2 \\ \phantom{2}x + 2y + \phantom{2}z = 3 \\ 2x + 3y + 2z = c \end{cases}$$
where $c$ is an arbitrary constant.

(a) Put the system in staircase form. You don't need to compute the RREF.
(b) Fill in the blanks: The first two planes meet in a line $L$. When $c = 5$ we have $\infty$ many solutions because the third plane _______ , but when $c = 6$ we have 0 solutions because the third plane _______ .
(c) Fill in the blank: It is impossible for the system to have exactly 1 solution because if we have one solution
$$x_1 \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + y_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + z_1 \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ c \end{pmatrix},$$
then we also have another solution _______ . [Hint: Change $x_1$ and $z_1$ somehow. The value of $c$ is irrelevant.]

**Problem 4.** Consider the following linear system:
$$\begin{cases} 0 + \phantom{2}x_2 + \phantom{x_3}0 + \phantom{4}x_4 - x_5 - 4x_6 = -1 \\ x_1 + 2x_2 - x_3 + 4x_4 - x_5 - 4x_6 = \phantom{-}3 \\ x_1 + 2x_2 - x_3 + 4x_4 + \phantom{x_5}0 - \phantom{4}x_6 = \phantom{-}5 \end{cases}$$

(a) Compute the RREF of the system.
(b) Write down the full solution in parametric form.

# Old HW3 Solutions

Problem 1:

(a) Let a, b, c, d be constants and consider the linear equation

$$ax + by + cz = d.$$

[This represents a plane in 3D]. Now suppose we have two points $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$ such that

- $ax_1 + by_1 + cz_1 = d$
- $ax_2 + by_2 + cz_2 = d$.

Then we must also have

$$a\left(\frac{x_1+x_2}{2}\right) + b\left(\frac{y_1+y_2}{2}\right) + c\left(\frac{z_1+z_2}{2}\right)$$

$$= \frac{1}{2}\left[a(x_1+x_2) + b(y_1+y_2) + c(z_1+z_2)\right]$$

$$= \frac{1}{2}\left[ax_1 + ax_2 + by_1 + by_2 + cz_1 + cz_2\right]$$

$$= \frac{1}{2}\left[(ax_1+by_1+cz_1)+(ax_2+by_2+cz_2)\right]$$

$$= \frac{1}{2}\left[d+d\right]$$

$$= \frac{1}{\cancel{2}}\left[\cancel{2}d\right] = d.$$

Hence the midpoint is also a solution.

[Geometrically, if two points lie on a plane in 3D then their midpoint also lies on the plane. This is why we say a plane is "flat".]

(b) If 25 hyperplanes in 12-dimensional space meet at two given points then they must also meet at

the midpoint of those two points.

The reason is the same as in part (a), i.e., because hyperplanes are "flat".

[We will give a quick algebraic proof later in the language of "matrix algebra".]

Problem 2: Consider the linear system

$$\begin{cases} x + y + z = 2 & (1) \\ x + 2y + z = 3 & (2) \\ x + 3y + 2z = 5 & (3) \end{cases}$$

(a) We write the system as an augmented matrix and then perform Gaussian elimination as follows:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & 2 & 1 & 3 \\ 1 & 3 & 2 & 5 \end{array}\right) \begin{array}{l} (1) \\ (2) \\ (3) \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & 1 & 3 \end{array}\right) \begin{array}{l} (1) \to (1) \\ (2) \to (2) - (1) \\ (3) \to (3) - (1) \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right) \begin{array}{l} (1) \to (1) \\ (2) \to (2) \\ (3) \to (3) - 2(2) \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right) \begin{array}{l} (1) \to (1) - (3) \\ (2) \to (2) \\ (3) \to (3) \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right) \quad \begin{array}{l} ① \to ① - ② \\ ② \to ② \\ ③ \to ③ \end{array}$$

Now we translate this back into a linear system

$$\begin{cases} x + 0 + 0 = 0 \\ 0 + y + 0 = 1 \\ 0 + 0 + z = 1 \end{cases}$$

(b) Row Picture : The three original planes intersect at the single point

$$(x, y, z) = (0, 1, 1).$$

(c) Column Picture : We can reach the target vector $(2, 3, 5)$ with the following linear combination :

$$0 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + 1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + 1 \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}.$$

[ Pictures omitted ]

Problem 3. Let $c$ be a constant and consider the linear system

$$\begin{cases} x + y + z = 2 & (1) \\ x + 2y + z = 3 & (2) \\ 2x + 3y + 2z = c & (3) \end{cases}$$

(a) To put the system in staircase form we first eliminate below the $x$ pivot to get

$$\begin{cases} x + y + z = 2 & (1) \to (1) \\ 0 + y + 0 = 1 & (2) \to (2) - (1) \\ 0 + y + 0 = c - 4 & (3) \to (3) - 2(1) \end{cases}$$

Then we eliminate below the $y$ pivot to get

$$\begin{cases} x + y + z = 2 & (1) \to (1) \\ 0 + y + 0 = 1 & (2) \to (2) \\ 0 + 0 + 0 = c - 5 & (3) \to (3) - (2) \end{cases}$$

Note that we obtain the equation $0 = c - 5$. If $c \neq 5$ then the system will have NO SOLUTION.

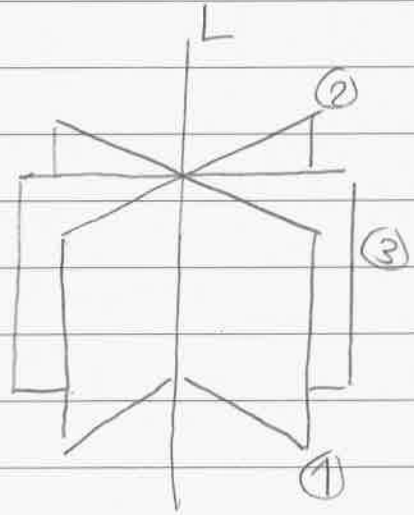
(b) The first two planes meet in a line
Call it $L$. When $c = 5$ then we have
$\infty$ many solutions because the third
plane

contains the line $L$,

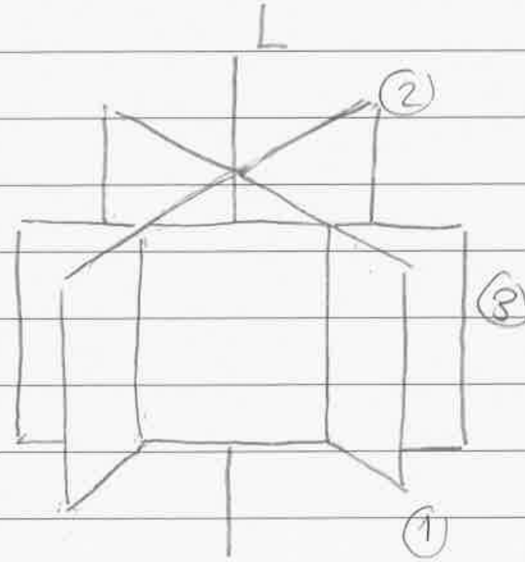
but when $c = 6$ (more generally for
any $c \neq 5$) we have $0$ solutions
because the third plane

does not intersect (i.e. is
parallel to) the line $L$.

Picture:




$c = 5$




$c \neq 5$.

(c) It is impossible for the system to have exactly 1 solution because if we have one solution

$$(*) \qquad x_1 \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + y_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + z_1 \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ c \end{pmatrix},$$

then we also have another solution,

$$(x_1 + k, y_1, z_1 - k) \text{ for any } k.$$

Proof: Assume $(*)$ is true. Then we have

$$(x_1 + k) \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + y_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + (z_1 - k) \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

$$= x_1 \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + y_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + z_1 \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + \cancel{k \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} - k \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}}$$

$$= \begin{pmatrix} 2 \\ 3 \\ c \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ c \end{pmatrix}.$$

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Problem 4. Consider the linear system

$$\begin{cases} 0 + x_2 + 0 + x_4 - x_5 - 4x_6 = -1 & ① \\ x_1 + 2x_2 - x_3 + 4x_4 - x_5 - 4x_6 = 3 & ② \\ x_1 + 2x_2 - x_3 + 4x_4 + 0 - x_6 = 5 & ③ \end{cases}$$

(a) We write the system as an augmented matrix and then perform Gaussian elimination to obtain the RREF

$$\left(\begin{array}{cccccc|c} 0 & 1 & 0 & 1 & -1 & -4 & -1 \\ 1 & 2 & -1 & 4 & -1 & -4 & 3 \\ 1 & 2 & -1 & 4 & 0 & -1 & 5 \end{array}\right) \begin{array}{c} ① \\ ② \\ ③ \end{array}$$

$$\left(\begin{array}{cccccc|c} 1 & 2 & -1 & 4 & -1 & -4 & 3 \\ 0 & 1 & 0 & 1 & -1 & -4 & -1 \\ 1 & 2 & -1 & 4 & 0 & -1 & 5 \end{array}\right) \begin{array}{c} ① \to ② \\ ② \to ① \\ ③ \to ③ \end{array}$$

REF

$$\left(\begin{array}{cccccc|c} 1 & 2 & -1 & 4 & -1 & -4 & 3 \\ 0 & 1 & 0 & 1 & -1 & -4 & -1 \\ 0 & 0 & 0 & 0 & 1 & 3 & 2 \end{array}\right) \begin{array}{c} ① \to ① \\ ② \to ② \\ ③ \to ③ - ① \end{array}$$

$$\left(\begin{array}{cccccc|c} 1 & 2 & -1 & 4 & 0 & -1 & 5 \\ 0 & 1 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 3 & 2 \end{array}\right) \begin{array}{c} ① \to ① + ③ \\ ② \to ② + ③ \\ ③ \to ③ \end{array}$$

RREF
$$\left(\begin{array}{cccccc|c} 1 & 0 & -1 & 2 & 0 & 1 & 3 \\ 0 & 1 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 3 & 2 \end{array}\right) \quad \begin{array}{l} ① \to ① - 2② \\ ② \to ② \\ ③ \to ③ \end{array}$$

Then we turn this back into a linear system.

$$\begin{cases} x_1 + 0 - x_3 + 2x_4 + 0 + x_6 = 3 \\ \quad x_2 + 0 + x_4 + 0 - x_6 = 1 \\ \qquad\qquad x_5 + 3x_6 = 2 \end{cases}$$

The pivot variables are $x_1, x_2, x_5$ and the free variables are $x_3, x_4, x_6$. Let's define parameters $r := x_3$, $s := x_4$, $t := x_6$. Then the solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 3 + r - 2s - t \\ 1 - s + t \\ r \\ s \\ 2 - 3t \\ t \end{pmatrix}$$

We get a better of the solution if we write it like this:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 2 \\ 0 \end{pmatrix} + r \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ -3 \\ 1 \end{pmatrix}$$

This is a 3-dimensional plane living in 6-dimensional space, which is what we expected because

$$\begin{array}{ccc} \#\text{variables} & - & \#\text{equations} \\ 6 & - & 3 \end{array} = 3 \checkmark$$