MTH 210, Spring 2016
HW 7 Solutions

Problem 1:

(a) Recall that the rotation matrix is

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Its characteristic equation is

$$0 = \det(R_\theta - \lambda I_2)$$

$$= \det\begin{pmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{pmatrix}$$

$$= (\cos\theta - \lambda)^2 + (\sin\theta)^2$$

$$= \lambda^2 - 2\cos\theta\,\lambda + \underbrace{(\cos\theta)^2 + (\sin\theta)^2}$$

$$= \lambda^2 - 2\cos\theta\,\lambda + 1.$$

and therefore its eigenvalues are

$$\lambda = \frac{2\cos\theta \pm \sqrt{4(\cos\theta)^2 - 4}}{}$$

$$= \frac{2\cos\theta \pm 2\sqrt{(\cos\theta)^2 - 1}}{2}$$

$$= \cos\theta \pm \sqrt{(\cos\theta)^2 - 1} .$$

Note that $(\cos\theta)^2 - 1 \leq 0$ so the eigenvalues will not be "real" unless

$$(\cos\theta)^2 - 1 = 0$$
$$(\cos\theta)^2 = 1$$
$$\cos\theta = \pm 1$$
$$\theta = 0^\circ \text{ or } 180^\circ .$$

When $\theta = 0^\circ$ the matrix $R_\theta = R_0 = I_2$ has the eigenvalue $\cos(0) = 1$. This makes sense because the identity matrix sends every vector $\vec{x}$ to itself:

$$I_2 \vec{x} = 1\vec{x} .$$

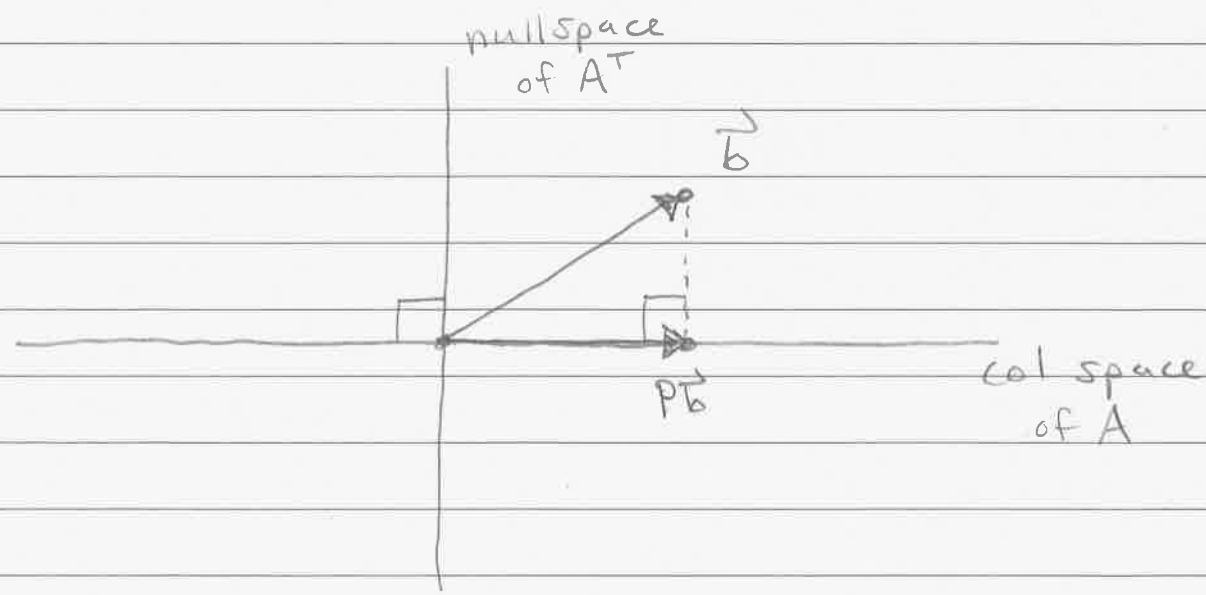
When $\theta = 180^\circ$ the matrix $R_{180^\circ} = -I_2$ has the eigenvalue $\cos(180^\circ) = -1$.

This makes sense because the matrix $-I_2$ sends every vector $\vec{x}$ to the negative of itself:

$$-I_2 \vec{x} = (-1)\vec{x}.$$

Apart from $\theta = 0°$ & $180°$, the rotation matrix R_θ will have NO (real) EIGENVECTORS.

(b) Let $P = A(A^TA)^{-1}A^T$ be the orthogonal projection onto the column space of A. We have discussed that the vectors $\vec{e}$ that are perpendicular to the column space of A satisfy $A^T\vec{e} = \vec{0}$, that is, they are in the "nullspace" of A^T. Here is a picture of the situation, looking at it "from the side":



In the picture I have drawn a general vector $\vec{b}$ and its projection $P\vec{b}$ into the column space.

Q: For which vectors $\vec{b}$ will $\vec{b}$ & $P\vec{b}$ be in the same direction, that is, which vectors $\vec{b}$ are eigenvectors for P?

From the picture it's clear that there are only two ways this can happen.

- If $\vec{b}$ is in the column space of A then we have $P\vec{b} = \vec{b} = 1\vec{b}$, so $\vec{b}$ is an eigenvector of eigenvalue 1.

- If $\vec{b}$ is in the nullspace of A^T then we have $P\vec{b} = \vec{0} = 0\vec{b}$, so $\vec{b}$ is an eigenvector of eigenvalue 0.

No other vector $\vec{b}$ is an eigenvector of P, so the only eigenvalues of P are

$$1 \ \& \ 0.$$

[Here's an alternative explanation: Let $\vec{x}$ be an eigenvector with $P\vec{x} = \lambda\vec{x}$. Then recall from HW6 that $P^2 = P$, so we have

$$\lambda\vec{x} = P\vec{x} = P^2\vec{x} = \lambda^2\vec{x}.$$

Since $\vec{x} \neq \vec{0}$ this implies that $\lambda = \lambda^2$ and hence $\lambda = 1$ or 0.]

(c) Now suppose P is some matrix whose eigenvalues are 1 & 0 (maybe P is the projection matrix from part (b)). This means that there exist nonzero vectors $\vec{x}$ & $\vec{y}$ such that

$$P\vec{x} = 1\vec{x} = \vec{x} \quad \& \quad P\vec{y} = 0\vec{y} = \vec{0}.$$

But then notice that

$$(2P - I)\vec{x} = 2P\vec{x} - I\vec{x} = 2\vec{x} - \vec{x} = \vec{x} = 1\vec{x}$$

&

$$(2P - I)\vec{y} = 2P\vec{y} - I\vec{y} = 2\vec{0} - \vec{y} = (-1)\vec{y},$$

which implies that 1 & -1 are eigenvalues of the matrix $(2P - I)$.

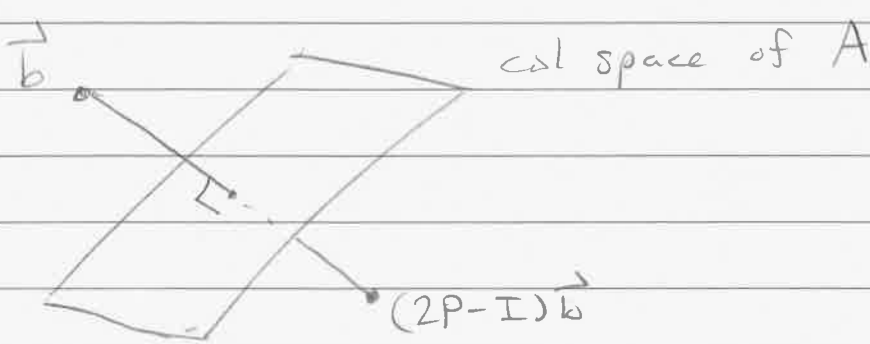
Could the matrix $(2P-I)$ have any other eigenvalues? Suppose there exists a nonzero vector $\vec{z} \neq \vec{0}$ such that $(2P-I)\vec{z} = \lambda\vec{z}$. Then we must have

$$(2P-I)\vec{z} = \lambda\vec{z}$$
$$2P\vec{z} - \vec{z} = \lambda\vec{z}$$
$$2P\vec{z} = (\lambda+1)\vec{z}$$

$$P\vec{z} = \left(\frac{\lambda+1}{2}\right)\vec{z},$$

which implies that $(\lambda+1)/2$ is an eigenvalue of P. Since the only eigenvalues of P are 1 & 0 we conclude that $\lambda = 1$ or -1. ///

Geometric Interpretation: If P is the orthogonal projection onto the column space of A then $(2P-I)$ is the orthogonal reflection across the column space



Note that performing a reflection twice is the same as performing it zero times, so that

$$(2P - I)(2P - I) = I$$

[This also says that the matrix $(2P - I)$ is equal to its own inverse.]

We could easily turn this into a proof that the eigenvalues λ of $(2P - I)$ must satisfy $\lambda^2 = 1$, hence $\lambda = 1$ or -1. ///

Problem 2:

(a) This was explained adequately in class so I won't do it again.

(b) Consider the matrix $T = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. Its characteristic equation is

$$0 = \det(T - \lambda I_2)$$

$$= \det \begin{pmatrix} 1-\lambda & 1 \\ 1 & 0-\lambda \end{pmatrix}$$

$$= (1-\lambda)(-\lambda) - 1$$
$$= \lambda^2 - \lambda - 1$$

$$\boxed{0 = \lambda^2 - \lambda - 1}$$

Using the quatratic formula gives us two real eigenvalues. We will call them

$$\varphi_1 = (1+\sqrt{5})/2 \quad \& \quad \varphi_2 = (1-\sqrt{5})/2$$
$$\approx 1.61 \qquad\qquad \approx -0.61$$

(c) According to (b) the equation

$$T\vec{x} = \varphi_1 \vec{x}$$
$$(T - \varphi_1 I_2)\vec{x} = \vec{0}$$

has a nonzero solution. Let's find it:

$$\begin{pmatrix} 1-\varphi_1 & 1 \\ 1 & -\varphi_1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 1-\varphi_1 & 1 & 0 \\ 1 & -\varphi_1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & -\varphi_1 & 0 \\ 1-\varphi_1 & 1 & 0 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cc|c} 1 & -\varphi_1 & 0 \\ 0 & 0 & 0 \end{array}\right) \longrightarrow \vec{x} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \varphi_1 y \\ y \end{pmatrix} = y\begin{pmatrix} \varphi_1 \\ 1 \end{pmatrix}.$$

We conclude that $T\begin{pmatrix}\varphi_1\\1\end{pmatrix} = \varphi_1\begin{pmatrix}\varphi_1\\1\end{pmatrix}$.

We also know that $(T-\varphi_2 I_2)\vec{x} = \vec{0}$ has a nonzero solution:

$$\left(\begin{array}{cc|c}1-\varphi_2 & 1 & 0\\ 1 & -\varphi_2 & 0\end{array}\right) \longrightarrow \begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \left(\begin{array}{cc|c}1 & -\varphi_2 & 0\\ 0 & 0 & 0\end{array}\right)\end{array}$$

$$\longrightarrow \quad \vec{x} = \begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}\varphi_2 y\\y\end{pmatrix} = y\begin{pmatrix}\varphi_2\\1\end{pmatrix}.$$

We conclude that

$$\boxed{T\begin{pmatrix}\varphi_1\\1\end{pmatrix} = \varphi_1\begin{pmatrix}\varphi_1\\1\end{pmatrix} \quad \& \quad T\begin{pmatrix}\varphi_2\\1\end{pmatrix} = \varphi_2\begin{pmatrix}\varphi_2\\1\end{pmatrix}}.$$

These boxed formulas tell us everything we need to know about Fibonacci numbers.

(c) Recall the initial condition $\vec{f}_0 = \begin{pmatrix}f_1\\f_0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$ leads to the equation

$$\begin{pmatrix}f_{n+1}\\f_n\end{pmatrix} = T^n\begin{pmatrix}1\\0\end{pmatrix}.$$

Thus we want to express (1,0) in terms of eigenvectors for T. That is, we want to solve for a & b in the following:

$$a\begin{pmatrix}\varphi_1\\1\end{pmatrix}+b\begin{pmatrix}\varphi_2\\1\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix}$$

$$\begin{pmatrix}\varphi_1 & \varphi_2\\1 & 1\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix}.$$

$$\left(\begin{array}{cc|c}\varphi_1 & \varphi_2 & 1\\1 & 1 & 0\end{array}\right)\begin{array}{l}R_1\\R_2\end{array}$$

$$\left(\begin{array}{cc|c}1 & 1 & 0\\\varphi_1 & \varphi_2 & 1\end{array}\right)\begin{array}{l}R_2\\R_1\end{array}$$

$$\left(\begin{array}{cc|c}1 & 1 & 0\\0 & \varphi_2-\varphi_1 & 1\end{array}\right)\begin{array}{l}R_1\\R_2-\varphi_1R_1\end{array}$$

[Now observe that $\varphi_2-\varphi_1=\frac{1-\sqrt{5}}{2}-\frac{1+\sqrt{5}}{2}=-\sqrt{5}$]

$$\left(\begin{array}{cc|c}1 & 1 & 0\\0 & 1 & -1/\sqrt{5}\end{array}\right)\begin{array}{l}R_1\\-1/\sqrt{5}\,R_2\end{array}$$

$$\left(\begin{array}{cc|c}1 & 0 & 1/\sqrt{5}\\0 & 1 & -1/\sqrt{5}\end{array}\right)\begin{array}{l}R_1-R_2\\R_2\end{array}$$

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We conclude that $a = 1/\sqrt{5}$ & $b = -1/\sqrt{5}$. Then finally we get

$$\begin{pmatrix} f_{n+1} \\ f_n \end{pmatrix} = T^n \left[\frac{1}{\sqrt{5}} \begin{pmatrix} \varphi_1 \\ 1 \end{pmatrix} - \frac{1}{\sqrt{5}} \begin{pmatrix} \varphi_2 \\ 1 \end{pmatrix} \right]$$

$$= \frac{1}{\sqrt{5}} T^n \begin{pmatrix} \varphi_1 \\ 1 \end{pmatrix} - \frac{1}{\sqrt{5}} T^n \begin{pmatrix} \varphi_2 \\ 1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{5}} \varphi_1^n \begin{pmatrix} \varphi_1 \\ 1 \end{pmatrix} - \frac{1}{\sqrt{5}} \varphi_2^n \begin{pmatrix} \varphi_2 \\ 1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{5}} \begin{pmatrix} \varphi_1^{n+1} - \varphi_2^{n+1} \\ \varphi_1^n - \varphi_2^n \end{pmatrix}$$

and hence the n^{th} Fibonacci number is

$$f_n = (\varphi_1^n - \varphi_2^n) / \sqrt{5}.$$

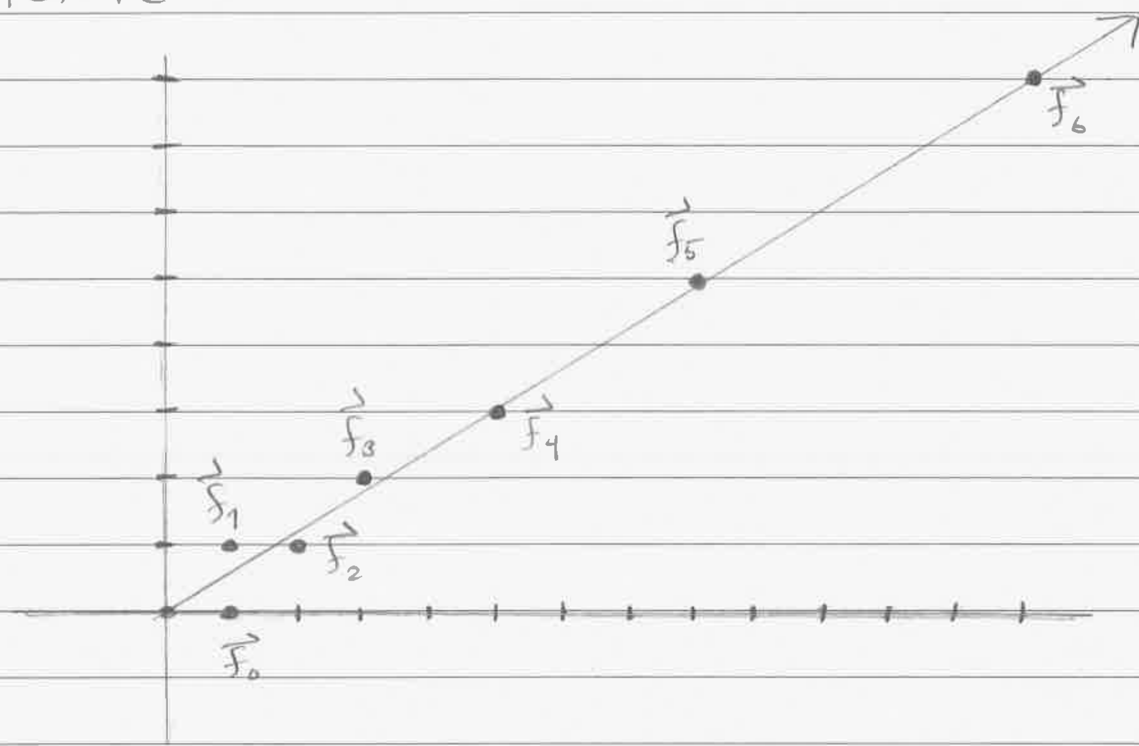
Strange but true!

(d) I will draw the two "eigenspaces"

$$t \begin{pmatrix} \varphi_1 \\ 1 \end{pmatrix} \quad \& \quad t \begin{pmatrix} \varphi_2 \\ 1 \end{pmatrix}$$

along with the first few vectors $\vec{f}_n = (f_{n+1}, f_n)$.

Picture:



Note that the points $\vec{f}_n$ are getting closer and closer to the line $t(\varphi_1, 1)$ while bouncing back in forth. This is not a surprise because

$$\varphi_2^n = (-0.61)^n \longrightarrow 0$$

as $n \to \infty$, while bouncing back and forth.