Math 210 F **Spring 2016**
Homework 6 **Drew Armstrong**

Problem 1. Projection Matrices (Continued from HW5.4). Recall that we call P a projection matrix if $P^T = P$ and $P^2 = P$.

(a) If A is any **rectangular** matrix such that $(A^TA)^{-1}$ exists, show that $P = A(A^TA)^{-1}A^T$ is a projection matrix. We will see in class that this is the matrix of the orthogonal projection onto the column space of A.

(b) If A is a **square** matrix such that A^{-1} exists, show that $P = A(A^TA)^{-1}A^T = I$. What does this mean? What space does this matrix project onto?

Problem 2. Projections in Three Dimensional Space. Consider the following vector $\vec{a}$ in $\mathbb{R}^3$ and the corresponding orthogonal plane:

$$\vec{a} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \qquad \text{and} \qquad x + 2y + 2z = 0.$$

(a) Use the formula from Problem 1 to find the 3×3 matrix P_1 that projects onto the line $t\vec{a}$. [Hint: Just let $A = \vec{a}$.]

(b) Use the matrix P_1 to project the vector $\vec{b} = (1, -1, 1)$ onto the line.

(c) Find two vectors in the plane $x + 2y + 2z = 0$ and then use the formula from Problem 1 to find the 3×3 matrix P_2 that projects onto the plane. [Hint: Let A be the 3×2 matrix whose columns are the two vectors that you found.]

(d) Use the matrix P_2 to project the vector $\vec{b} = (1, -1, 1)$ onto the plane.

(e) Finally, check that $P_1 + P_2 = I$. Does this surprise you?

Problem 3. Minimizing the Distance from a Point to a Plane. What linear combination of $(1, 2, -1)$ and $(1, 0, 1)$ is closest to $(3, -1, -1)$? [Hint: Problem 2(e) might suggest a shortcut.]

Problem 4. Average, Variance, Standard Deviation. These concepts from statistics are a very special case of least squares approximation.

(a) Find the equation of the **horizontal line** $C + 0t = b$ that is the best fit for the data

$$\begin{pmatrix} t \\ b \end{pmatrix} = \begin{pmatrix} t_1 \\ 1 \end{pmatrix}, \begin{pmatrix} t_2 \\ 4 \end{pmatrix}, \begin{pmatrix} t_3 \\ 7 \end{pmatrix}, \begin{pmatrix} t_4 \\ 2 \end{pmatrix}.$$

[Hint: I didn't tell you values t_1, t_2, t_3, t_4 because they don't matter. You are trying to solve the unsolvable system of equations $C = 1$, $C = 4$, $C = 7$ and $C = 2$. Write this system as $A\vec{x} = \vec{b}$ and then solve the normal equation $A^TA\hat{x} = A^T\vec{b}$ instead.]

(b) More generally, consider any $m \times 1$ vector $\vec{b} = (b_1, b_2, \ldots, b_m)$. Compute the average of the entries b_i by projecting the vector $\vec{b}$ onto the line through $\vec{a} = (1, 1, \ldots, 1)$. That is, solve for the average $\hat{x}$ in the normal equation $\vec{a}^T\vec{a}\hat{x} = \vec{a}^T\vec{b}$.

(c) Continuing from part (b), compute the "error vector" $\vec{e} = \vec{b} - \vec{a}\hat{x}$. Then compute the variance $\|\vec{e}\|^2$ and the standard deviation $\|\vec{e}\|$.

Problem 5. Best Fit Line. Find the equation $C + tD = b$ of the best fit line for the data

$$\begin{pmatrix} t \\ b \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix},$$

using the following steps:

(a) Write down the matrix equation $A\vec{x} = \vec{b}$ that **would be** true if all four points were on the same line $C + tD = b$. This equation has no solution.
(b) Now write down the normal equation $A^T A\hat{x} = A^T\vec{b}$ and solve it to find the least squares approximation $\hat{x} = (C, D)$.
(c) Compute the error vector $\vec{e} = \vec{b} - A\hat{x}$.
(d) Finally, draw the four data points along with their best fit line. Label the vertical errors with the entries of the error vector $\vec{e}$.